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so that on the Mercator chart the curves $X =$ constant and $Y =$ constant are Sumner lines; and these are the projections of small circles drawn through the tangent lines of the meridian and equator at the pole of the stereographic projection

28. In the ordinary stereographic or Mercator chart of the two hemispheres, 24 meridians are drawn at intervals of 15° of longitude from Greenwich, to correspond to the 24 hours of the day; and these 24 standard meridians are now used to define the *standard time* at any point on the Earth; the standard time being the mean solar time at the nearest standard meridians, changing by a whole hour when the traveller has gone more than $7\frac{1}{2}°$ of longitude, east or west of a standard meridian.

Thus between longitude $7° \frac{1}{2}$ E. and W. of Greenwich, the standard time is Greenwich time; and travelling westward the meridian 15° W. defines Ocean or Azores time, 30° W. Colonial time, 45° W. New York time, 60° W. Chicago time, and so on.

29. In the stereographic projection of the Eastern and Western hemispheres, the poles are taken on the equator, but when we wish to represent stereographically the hemispheres of greatest land and of greatest water, the Continental and the Oceanic hemispheres, or to represent the stereographic chart of the sky for a particular place, say Greenwich or Paris, the pole of projection will be no longer in the equator, but will be the antipole of the corresponding hemisphere, or of the horizon of the place.

30. Prof. Cayley has shown that the meridians and parallels of the terrestrial sphere will in every case project stereographically into a system of dipolar circles, and that the algebraic transformation of the coordinates from one pole to any other in the stereographic system is always a *homographic* transformation, which may be represented geometrically by the finite rotation to be given to the sphere to bring the new pole on to the diameter of the sphere which is perpendicular to the plane of the stereographic chart (*Transactions of the Cambridge Philosophical Society, XIII, The Polyhedron Functions*: Klein's *Lectures on the Ikosahedron.* translated by G. G. Morrice. Chap. II).

Any point whose coordinates on the plane of the map were originally X, Y is projected back on to the terrestrial sphere

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touching the plane; the sphere is rotated about any diameter through a finite angle β; and then the point on the sphere is again projected stereographically on the plane, its coordinates being X', Y', suppose.

Then denoting $X+iY$ by z and $X'+iY'$ by z', Prof. Cayley has shown that z and z' are always connected by a homographic relation of the form

$$z' = \frac{Az+B}{Cz+D},$$

and that

$$\cos\tfrac{1}{2}\beta = \frac{A+D}{\sqrt{(AD-BC)}}.$$

31. Suppose for instance we change from a pole on the equator O to a new pole Z on the meridian of O; the meridians and parallels in the two maps will be the same system of dipolar circles, and the meridians retain the same numbering in the two systems; but the parallels must be differently numbered.

Draw the double ordinate eZw through Z of the circle on the line NS, joining the poles N and S as diameter, and on this double ordinate as diameter describe a circle; this will represent the horizon of the plane having Z in the zenith.

Join wN, wS, cutting the horizon circle in Q and Q'; then QZQ' will be a diameter, and the angle NZQ will be the latitude of Z.

The meridians will be the circles through the poles N and S, the same as for the direct projection with O as pole.

To draw the parallel of latitude θ, lay off the angles QZa and QZb each equal to the colatitude, and join wa, wb cutting SN in c and d; then the circle on cd as diameter will be the required parallel of latitude θ on the oblique stereographic projection, with Z as pole (*Encyclopædia Britannica*, *Geography*, by Col. A. R. Clarke).

32. Denoting by β the latitude of Z, and putting $ON=OS=c$, then $wZ=c\cos\beta$; and

$$Zwc=\tfrac{1}{2}(\theta-\beta), \quad Zwd=\tfrac{1}{2}(\pi-\theta-\beta);$$

so that
$$cd = c\cos\beta\{\cot\tfrac{1}{2}(\theta+\beta)-\tan\tfrac{1}{2}(\theta-\beta)\}$$
$$= \frac{2c\cos\beta\cos\theta}{\sin\beta+\sin\theta}.$$

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But if θ' denotes the latitude on the direct projection with O as pole of the same parallel on cd as diameter, then

$$cd = 2c \cot \theta',$$

so that
$$\cot \theta' = \frac{\cos\beta \cos\theta}{\sin\beta + \sin\theta},$$

$$\operatorname{cosec} \theta' = \frac{1 + \sin\beta \sin\theta}{\sin\beta + \sin\theta},$$

$$\tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta') = \frac{1 - \sin\theta'}{1 + \sin\theta'}$$

$$= \frac{1 - \sin\beta}{1 + \sin\beta} \frac{1 - \sin\theta}{1 + \sin\theta}$$

$$= \tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\beta) \tan^2(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta),$$

or
$$\sin\theta' = \frac{\sin\beta + \sin\theta}{1 + \sin\beta \sin\theta};$$

this is the relation connecting the true anomaly $\frac{1}{2}\pi - \theta'$ from aphelion with the excentric anomaly $\frac{1}{2}\pi - \theta$ in an elliptic orbit of excentricity $\sin\beta$.

33. This relation is simplified on transference to Mercator's projection, by putting

$$\beta = \operatorname{gd} b/c, \quad \theta = \operatorname{gd} y/c, \quad \theta' = \operatorname{gd} y'/c;$$

then, since
$$\tan\theta' = \frac{\sin\beta + \sin\theta}{\cos\beta \cos\theta},$$

therefore
$$\sinh y'/c = \frac{\tanh b/c + \tanh y/c}{\operatorname{sech} b/a \operatorname{sech} y/c}$$

$$= \sinh(y + b)/c,$$

so that
$$y'/c = (y + b)/c;$$

and therefore
$$(X' + iY')/C = \tan\tfrac{1}{2}(x + i.y + b)/c.$$

This is otherwise obvious from the fact that the finite rotation of the sphere which changes from the direct to the oblique stereographic projection is through an angle β, about an axis perpendicular to the meridian, and defined in Klein's notation (*Ikosahedron*, p. 36), by $\xi = 1$, $\eta = 0$, $\zeta = 0$, $a = \sin\frac{1}{2}\beta$, $b = 0$, $c = 0$, $d = \cos\frac{1}{2}\beta$; so that

$$z' = \frac{z \cos\frac{1}{2}\beta + i \sin\frac{1}{2}\beta}{iz \sin\frac{1}{2}\beta - \cos\frac{1}{2}\beta}$$

$$= \frac{z + i \tan\frac{1}{2}\beta}{1 - iz \tan\frac{1}{2}\beta};$$

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and then if $$z = \tan\tfrac{1}{2}(x+iy)/c,$$

$$i\tan\tfrac{1}{2}\beta = i\tanh\tfrac{1}{2}b/c = \tan\tfrac{1}{2}ib/c,$$

$$z' = \tan\tfrac{1}{2}(x+i.y+b)/c,$$

as above.

34. The equation of a great circle on Mercator's Chart being always of the form

$$\sinh y/c = A\sin x/c + B\cos x/c,$$

it follows that the equation of the great circle through the two points whose coordinates on Mercator's Chart are (a, b), (a', b'), is

$$\sin(a-a')/c\sinh y/c = \sinh b/c\sin(x-a)/c - \sinh b'/c\sin(x-a')/c;$$

for this equation is satisfied by the coordinates of the two given points.

If X, Y are the coordinates of the corresponding point on the stereographic chart

$$\tan x/c = \frac{2CX}{C^2-X^2-Y^2},$$

so that $$\sin x/c = \frac{2CX}{D},\quad \cos x/c = \frac{C^2-X^2-Y^2}{D},$$

where $$D^2 = (C^2-X^2-Y^2)^2 + 4C^2X^2;$$

also $$\tanh y/c = \frac{2CY}{C^2+X^2+Y^2},$$

so that $$\sinh y/c = \frac{2CY}{D}.$$

Therefore the equation of the great circle on the stereographic chart passing through two points whose longitude and latitude are (α, β), (α', β'), is

$$2CY\sin(\alpha-\alpha') = (\tan\beta\cos\alpha' - \tan\beta'\cos\alpha)\,2CX$$
$$-(\tan\beta\sin\alpha' - \tan\beta'\sin\alpha)\,(C^2-X^2-Y^2).$$

35. The general equation of a small circle on the sphere of angular radius γ and pole at latitude α and longitude β is

$$\cos\gamma = \sin\alpha\sin\theta + \cos\alpha\cos\theta\cos(\phi-\beta).$$

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The corresponding line on Mercator's Chart is obtained by writing x/c for ϕ and $\mathrm{gd}\, g/c$ for θ; or $\tanh y/c$ for $\sin\theta$ and $\mathrm{sech}\, y/c$ for $\cos\theta$; and its equation is therefore

$$\cos\gamma \cosh y/c = \sin\alpha \sinh y/c + \cos\alpha \cos(x/c - \beta).$$

On the stereographic projection, we must put

$$\sin\phi = 2CX/D,\ \cos\phi = (C^2 - X^2 - Y^2)/D;$$

and $\sin\theta = 2CY/(C^2 + X^2 + Y^2)$, $\cos\theta = D/(C^2 + X^2 + Y^2)$;

and the equation of the small circle is therefore

$$(C^2 + X^2 + Y^2)\cos\gamma$$
$$= 2CY\sin\alpha + (C^2 - X^2 - Y^2)\cos\alpha\cos\beta + 2CX\cos\alpha\sin\beta.$$

ON WÖHLER'S EXPERIMENTS ON ALTERNATING STRESS.

By *Karl Pearson.*

On rapid changes of Stress.

1. It has long been known that elastic bodies subjected to long repeated vibrations or to rapid changes of stress ultimately *appear* to break-down under loads far less than the statical breaking loads. The practical importance of this apparent 'wearing out' or 'fatigue' of materials has led to the interesting experiments of Fairbairn, Wöhler and Bauschinger. The careful results of Wöhler have caused the adoption by many writers on the strength of materials of certain laws termed *Wöhler's Laws*, for the safe limit of load in materials subjected to vibration. These laws indicate that those who adopt them believe that a molecular change takes place in a material subjected to repeated stress, a change which would not take place in the same material subjected to a continuous stress of equal magnitude. The assumption of this molecular change seemed at first justified by the apparently 'crystalline fracture' of a 'fibrous' material. A change from crystalline to fibrous structure was supposed to take place owing to vibrational action in wrought-iron axles, chains, &c. Although this view was rejected from the first by some engineers of standing, notably George Stephenson, it yet appears to have met with the general acceptance of practical men till the accurate experiments of Kirkaldy showed that the same bar might be broken at will within the same inch of its length with either a 'crystalline' or a 'fibrous' fracture. Whether a

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molecular change differing from anything produced by statical stress does really occur under repeated changes of stress may be still an open question, but the support this hypothesis derived from the distinction between 'crystalline' and 'fibrous' fracture is, I think, illusory.

The opinion which has been forced upon me after some study of the writers on this subject is opposed to the belief that there is any distinct molecular change, characteristic of vibration or repeated loading. I am inclined to think that, in every case where a load less than the statical load destroys a material and this destruction is attributed to molecular change, it will be found that the kinetics of the problem really produce a stress in the body far above the limit of stress produced by a statical load of the same magnitude, and that this occurs *even in those cases in which there is nothing of the nature of impulse.* So far from it being possible to state any general law of repeated loading, I believe that each individual case must be treated on its own merits, and the proper limit of safe loading calculated from the maximum value of the kinematic stress. The fact that a bar after several hundred thousand changes of load breaks down must be sought in the kinetics of the particular problem, and these I hold will invariably show that a stress has really been reached at some point of the material, which exceeds the safe statical limit. This stress lasts only for a very short time, but continuously repeated ultimately destroys the material, just as the same stress applied statically would have done.

The fact that the kinetic stress exceeds the statical stress is due to the method of loading and to the rate of change of stress. It may arise from accumulations of stress due to wave vibrations in the piece of material, it may arise from a practically impulsive action of the load, or it may arise from the inertia of adjacent bodies connected with the system of loading. Each case requires separate investigation. The case of impulsive and repeated torsional loading I have investigated elsewhere. My attention has, however, recently been redrawn to the subject of continuous non-impulsive change of stress by a visit to the *City and Guilds' Central Institution*, where Professor Unwin kindly showed me his laboratory apparatus for exhibiting such change of stress at work. He told me that in this case about half the statical load was sufficient after a considerable number of changes (several millions) to destroy the bars experimented on.

Professor Unwin's apparatus was so much simpler than Wöhler's complex arrangement of levers that it seemed to

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me that the analysis of it could be easily worked out. It consisted practically of a bar of circular cross-section built in at one end and loaded with a weight at the free end, the bar being caused to rotate about its axis by a simple mechanical contrivance. In this case there did not appear to be any possibility of impulsive action, hence either the repeated change of stress produced a molecular change or there was an accumulation of stress due to waves along the bar, or again the inertia of the load was important. I soon found that at the rate at which the *short* bar of the experiment was rotated there was no sensible accumulation of stress by wave motion, so that, if my view were the correct one, the 'fatigue' must be due to the inertia of the load. This, I think, is shown to be *possible* by the investigation which follows in Case (i).

Professor Unwin drew my attention to the fact that Wöhler had replaced the suspended weight by a spring, but on examining the description Wöhler gives of his apparatus, it seems to me that his levers were sufficiently cumbersome to produce a similar if not as great an effect as the inertia of the weight in Professor Unwin's apparatus. It was not easy to work out the effect of Wöhler's levers in the case of his flexure experiments, but I have attempted to do this in the case of the like apparatus for continuous change from positive to negative tensile stress. In this case the inertia of the apparatus turns out to be of first-class importance. This is the subject matter of my Case (ii). Further I shew in Case (iii) that, unless the bar were of very great length, a mere change of stress without any inertia of loading would not allow of the accumulation of stress due to the reflection of the vibrations reaching any sensible magnitude.

My only excuse for the elementary analysis of these problems must be their practical importance. If the view that a molecular change does take place in the material be correct (which means that for safety the kinetic load must be much less than the statical), then I believe that the safe load must be made still less than the value due to this change in any apparatus like Wöhler's or Professor Unwin's, where a really sensible effect is produced by inertia. There would thus be two things to allow for, molecular change and inertia. Until it is shown that the latter is not sufficient to explain the results arrived at I must continue to question the reality of the assumed 'molecular change.'

In Cases (i) and (ii) I have not taken into account the initial state of stress on starting the machinery that produces the changes of stress. It is difficult to measure the exact

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influence of the initial stress, but in Case (iii), as well as in Case (i) as far as its value to the first harmonics is concerned, the effect is inappreciable. This is no more than we should expect, for some hundred repeated changes of stress must rapidly produce a condition of steady motion.

2. Case (i). *Continuous change from positive to negative flexure.*

Suppose a beam or rod built-in at one end and free at the other to be loaded with a weight P at the free end. If Eak_1^2 be the flexural rigidity of the beam in the plane of loading, l its length, b_1, b_2 the distances of the 'extreme fibres' in tension and conpression respectively from the 'neutral axis,' then the maximum positive and negative tractions are given respectively by;

$$T_1 = \frac{P}{a}\frac{lb_1}{k_1^2}, \text{ and } T_2 = \frac{P}{a}\frac{lb_2}{k_1^2} \quad \ldots\ldots\ldots\ldots \text{(i)}.$$

These measure the statical strength of the beam, and if T_1, T_2 be less than the tensile and compressive elastic limits, no damage will be done to the bar by this statical loading, however long continued.

Now Wöhler caused the bar to rotate about its longitudinal axis, so that the flexure was alternately positive and negative, and he found that if P were even less than the statical safe load, the bar was still ultimately destroyed by a definite number of revolutions. I propose in the present paper to investigate what is the proper limit P' for the load P, that the maximum kinetic stress shall not exceed the elastic limit.

3. The effect produced by the rotatory inertia of the bar will be small, and we may safely neglect it to a first approximation. We may then suppose the load P' to rotate, and the bar to be fixed. Let P' make n rotations in a second, and let ω be its spin round the axis, so that $2\pi n = \omega$. We may take for definiteness a principal axis of the cross-section, supposed uniform, to be vertical, and let P' make an angle ωt with this principal axis of the terminal cross-section. Then if Eak_1^2 be the rigidity in the vertical, Eak_2^2 the rigidity in the horizontal plane, we may treat the flexure in these planes independently, and due respectively to the shears in those directions produced by the weight P'. It will be sufficient to consider the flexure in the plane for which the rigidity is least, and we will suppose this the vertical. In the case of a cross-section with inertial isotropy $k_1 = k_2$, and our results will hold for any plane.

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The equation for the vibrations of the bar will be

$$\Delta \frac{d^2z}{dt^2} + Ek_1^2 \frac{d^4z}{dx^4} = 0 \quad \text{.....(ii).}$$

where z is the vertical shift of a point on the axis of the bar distant x from the built-in end, and Δ is the density of the material. Let $p^2 = \Delta/(Ek_1^2)$, or

$$p^2 \frac{d^2z}{dt^2} + \frac{d^4z}{dx^4} = 0 \quad \text{.....(iii).}$$

At the built-in ends

$$z = 0, \quad \frac{dz}{dx} = 0 \quad \text{.....(iv);}$$

at the free end the bending-moment is zero, or

$$Eak_1^2 \frac{d^2z}{dx^2} = 0 \quad \text{.....(v),}$$

and the shear $= F$, the force produced by P', or

$$Eak_1^2 \frac{d^3z}{dx^3} = -F.$$

But for the motion P' we have

$$\frac{P'}{g} \frac{d^2z}{dt^2} = F - P' \cos \omega t,$$

since gravity makes an angle ωt with the plane zx.

Hence we have, when $x = l$,

$$Eak_1^2 \frac{d^3z}{dx^3} = -P' \left(\cos \omega t + \frac{1}{g} \frac{d^2z}{dt^2} \right) \quad \text{.....(vi).}$$

Assume $z = z_1 + z_2$ where z_1 shall be the forced vibration due to the spin, and z_2 correspond to the free vibrations in period, z_1 will then be chosen to give the right-hand term in (vi), and satisfy otherwise equations (ii)—(v), while z_2 satisfies equations (ii)—(v) and (vi) with P' put equal to zero.

Let us first find z_1. Assume

$$z_1 = u_1 \cos \omega t \quad \text{.....(vii).}$$

Then from (iii) we have

$$\omega^2 p^2 u_1 = \frac{d^4 u_1}{dx^4} \quad \text{.....(viii),}$$

whence regarding (iv) we find

$$u_1 = A \{\sin \sqrt{(p\omega)}\, x - \sinh \sqrt{(p\omega)}\, x\} + B \{\cos \sqrt{(p\omega)}\, x - \cosh \sqrt{(p\omega)}\, x\} \quad \text{.....(viii),}$$

where A and B are constants.

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Substituting in (v) and writing

$$\sqrt{(p\omega)}\, l = \lambda \quad \text{.....(ix)},$$

we find

$$A(\sin\lambda + \sinh\lambda) + B(\cos\lambda + \cosh\lambda) = 0 \quad \text{.....(x)}.$$

Substituting in (vi), and writing

$$Q = P' / \{Eak_1^2 (p\omega)^{\frac{3}{2}}\} \quad \text{.....(xi)}.$$

we have

$$A(\cos\lambda + \cosh\lambda) - B(\sin\lambda - \sinh\lambda) = Q\left\{1 - \frac{\omega^2}{g} A(\sin\lambda - \sinh\lambda) - \frac{\omega^2}{g} B(\cos\lambda - \cosh\lambda)\right\} \quad \text{.....(xii)}.$$

Putting $\sin\lambda + \sinh\lambda = S$, $\sin\lambda - \sinh\lambda = S'$,

$\cos\lambda + \cosh\lambda = C$, $\cos\lambda - \cosh\lambda = C'$,

we have from (x) and (xii):

$$AS + BC = 0$$

$$A\left\{C^2 + SS' + \frac{Q\omega^2}{g}(S'C + SC')\right\} = QC.$$

Whence we find

$$\left.\begin{aligned} A &= \tfrac{1}{2} \frac{Q(\cos\lambda + \cosh\lambda)}{1 + \cos\lambda\cosh\lambda + \frac{Q\omega^2}{2g}(\sin 2\lambda - \sinh 2\lambda)} \\ B &= -\tfrac{1}{2} \frac{Q(\sin\lambda + \sinh\lambda)}{1 + \cos\lambda\cosh\lambda + \frac{Q\omega^2}{2g}(\sin 2\lambda - \sinh 2\lambda)} \end{aligned}\right\} \quad \text{.....(xiii)}.$$

Thus z_1 as given by (vii) is fully determined.

z_2 will be of the form

$$u_2 \cos(nt + \beta),$$

where

$$u_2 = A'\{\sin\sqrt{(pn)}x - \sinh\sqrt{(pn)}x\} + B'\{\cos\sqrt{(pn)}x - \cosh\sqrt{(pn)}x\},$$

while

$$A'\{\sin\sqrt{(pn)}l + \sinh\sqrt{(pn)}l\} + B'\{\cos\sqrt{(pn)}l + \cosh\sqrt{(pn)}l\} = 0$$

and n is a root of the equation

$$\cos\sqrt{(pn)}l \,.\, \cosh\sqrt{(pn)}l + 1 = 0.$$

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A' and β will then have to be determined by the initial conditions. If the bar start from rest at time $t=0$, we have $\beta=0$, and u_1+u_2 equals the initial deflection at x, if this be $f(x)$ we have to determine A'

$$f(x)-u_1=u_2.$$

I have worked out the value of A' for the first two harmonics, for one or two numerical examples, supposing the bar to start from rest in the position of statical strain due to P'. I find the terms in the stress (T_1) due to u_2 in these cases small as compared with those due to u_1, and this is natural. For as I have remarked the exact position of the bar at starting cannot after a considerable number of rotations really be of sensible importance in practise, *i.e.* the steady motion set up must be really independent of the exact condition at starting. I shall therefore confine my attention to the value of u_1.

4. Now the greatest positive and negative tractions will be given by

$$T_1=Eb\frac{d^2z_1}{dx^2},\quad T_2=Eb\frac{d^2z_1}{dx^2},\ \ldots\ldots\ldots\text{(xiv)},$$

where b is the greater of b_1 and b_2, and the maximum value is given to the differential. This will be reached when $\cos\omega t=1$ and thus d^2z_1/dx^2 may be replaced by d^2u_1/dx^2, which reaches its maximum for $x=0$, and then equals

$$-2p\omega B;$$

whence from (xiii) we have for T_1 or T_2:

$$T=\frac{Ebp\omega Q(\sin\lambda+\sinh\lambda)}{1+\cos\lambda\cosh\lambda+\dfrac{Q\omega^2}{2g}(\sin2\lambda-\sinh2\lambda)}.$$

Substituting for Q from (xi) and remembering (i), we have

$$P\left\{\lambda(1+\cos\lambda\cosh\lambda)+\frac{\lambda Q\omega^2}{2g}(\sin2\lambda-\sinh2\lambda)\right\}$$
$$=P'(\sin\lambda+\sinh\lambda).$$

But
$$\frac{\lambda Q\omega^2}{2g}=\frac{P'}{Ea}\frac{\omega^2l}{g}\left(\frac{l}{k_1}\right)^2\times\frac{1}{2\lambda^2}.$$

Hence

$$P'=\frac{P\lambda(1+\cos\lambda\cosh\lambda)}{\sin\lambda+\sinh\lambda+\dfrac{P}{Ea}\dfrac{\omega^2l}{g}\left(\dfrac{l}{k_1}\right)^2\dfrac{\sinh2\lambda-\sin2\lambda}{2\lambda^2}}\ \ldots\ldots\text{(xv)}.$$

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This result shews us that the kinetic safe load is very far from being of the same form as the statical safe load. This is partly due to the vibratory accumulation of stress, partly to the inertia of P'.

5. Suppose the load at the free terminal not to be produced by a suspended load but by a weightless spring, then we have

$$P' = \frac{P\lambda\,(1+\cos\lambda\cosh\lambda)}{\sin\lambda+\sinh\lambda} \quad \text{.....(xvi).}$$

If λ be very small, this may be written, W being the weight of the bar,

$$P' = P\frac{1}{1+\frac{11}{120}\frac{W}{E\alpha}\frac{\omega^2 l}{g}\left(\frac{l}{k_1}\right)^2} \quad \text{.....(xvii).}$$

If λ be very small and we include on the other hand the inertia of P, we find, from (xv),

$$P' = \frac{P}{1+\frac{2}{3}\frac{P}{E\alpha}\frac{\omega^2 l}{g}\left(\frac{l}{k_1}\right)^2} \quad \text{.....(xviii).}$$

Hence we see that for a small λ, or a slow spin in a short rod, the inertia of the load is far more important than the mere vibratory accumulation of stress, for P the safe statical load will always be much greater than W the weight of the bar.

6. We may now apply these results to one or two numerical examples.

Example (*a*). Suppose we take a bar of steel 16″ long and of circular cross-section with a diameter $\frac{1}{2}$″. Its stretch-modulus may be taken at 30,000,000 lbs. per square inch. We will suppose it to make three revolutions a second.

We have first to calculate λ,

$$\lambda = \sqrt{(p\omega)}\,l = \sqrt{\left(\frac{\omega l}{v}\,\frac{l}{k_1}\right)},$$

where v is the velocity of sound in steel = about 16000 feet per second,

$$2\pi/\omega = \tfrac{1}{3}, \quad k_1 = \tfrac{1}{8}''.$$

Hence $$\lambda = \sqrt{\left(\frac{6\pi\times 12\times 8}{16000}\right)} = \text{about } \tfrac{1}{3}.$$

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Hence, if we neglect λ^4 or terms of the order $\frac{1}{81}$, we may use formula (xviii).

We have first to calculate P from equation (i). The tensile strength of the steel might be taken as 100,000 lbs., so that we may safely consider 60,000 lbs. as a safe stress in flexure. Thus we find

$$60{,}000 \times \frac{k_1^2}{lb_1} = \frac{P}{a},$$

or
$$\frac{P}{a} = \frac{60{,}000}{64 \times 16 \times \frac{1}{4}} = \frac{60{,}000}{256}.$$

Thus

$$\tfrac{2}{3}\frac{P}{Ea}\frac{\omega^2 l}{g}\left(\frac{l}{k_1}\right)^2 = \tfrac{2}{3}\frac{60{,}000 \times 36 \times \pi^2 \times 16 \times 8 \times 16 \times 8}{64 \times 4 \times 30{,}000{,}000 \times 32}$$

$$= \frac{96\pi^2}{1000} = \frac{96}{100} = \cdot 96 \text{ nearly.}$$

Thus
$$P' = \frac{P}{1 \cdot 96},$$

or, the load which attached to the end of the rotating bar would ultimately destroy it *is about half the statical load which would produce the same effect.**

Example (*b*). Suppose P' applied by a weightless spring. We have now to use (xvii) as applied to the steel bar of Example (*a*). We must first calculate W. Supposing steel to weigh 486 lbs. per cubic foot, we have

$$W = \frac{a}{144} \times \frac{16}{12} \times 486 \text{ lbs.},$$

and
$$\tfrac{11}{120}\frac{W}{Ea}\frac{\omega^2 l}{g}\left(\frac{l}{k_1}\right)^2 = \frac{11 \times 16 \times 486 \times 36\pi^2 \times 8 \times 16 \times 8 \times 16}{120 \times 144 \times 12 \times 30{,}000{,}000 \times 32}$$

$$= \cdot 0025 \text{ nearly.}$$

Thus
$$P' = \frac{P}{1 \cdot 0025}$$

$$= P \text{ for all practical purposes.}$$

Thus we have the following conclusion, which ought to be capable of experimental verification: *If a steel bar such as the above rotating three times a second be subjected to a suspended load P', P' will ultimately destroy the bar if it be half the limiting*

* The bar in this example would I think *roughly* approximate to the bar in Professor Unwin's machine when I saw it.

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statical load, but if P' can be applied by a weightless spring or some system without inertia, it may be taken equal to the statical load before it will destroy the bar.

Example (c). Let us now consider an example of a wrought-iron bar of somewhat larger dimensions. Say, a bar 4′ long and of 2″ diameter, the modulus of which is 24,000,000 lbs. per square inch, and suppose it to make two revolutions a second, and weigh 450 lbs. per cubic foot.

Here it will be found that λ is again small and that we may use equation (xviii). If we suppose the maximum stress that wrought iron will bear without ultimate damage to be 30,000 lbs. per square inch, we have to find P,

$$P \times l = \frac{30{,}000\alpha \times k^2}{b_1}.$$

Whence

$$\tfrac{2}{3}\frac{P}{E\alpha}\frac{\omega^2 l}{g}\left(\frac{l}{k}\right)^2 = \tfrac{2}{3}\frac{30{,}000\alpha k^2 \times l^2}{24{,}000{,}000\alpha \times b_1 \times l \times k^2} \times \frac{16\pi^2 l}{g}$$

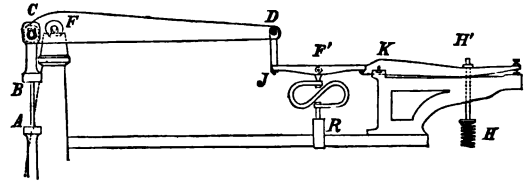
$$= \tfrac{2}{3}\frac{30{,}000}{24{,}000{,}000} \times \frac{l}{b_1} \times \tfrac{1}{2}(\pi^2 l) = \tfrac{1}{3} \text{ nearly.}$$

Thus we find $P' = \frac{1}{3}P$,

or only $\frac{1}{3}$ of the statical load could be applied in this fashion to a bar of the dimensions selected.

7. Case (ii). *To find the effect of the inertia of the lever in Wöhler's apparatus for gradual change of tractive stress.*

We may reduce Wöhler's apparatus to its simplest form in the following diagrammatic representation.



AB is the bar subjected to the changes of tractive stress and fixed at A. At B the bar is fixed to a lever CD turning about a fulcrum close to C. The end D of the lever is attached to a smaller lever JK which turns about the fulcrum F' supported by a stiff spring from the point R, which is caused to rapidly oscillate up and down, so that the spring dynamometer HH' is alternately extended and compressed. The force thus produced is carried through the levers $H'K$, KJ and

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DC to C, where it produces alternate tensile and compressive stress. In Wöhler's apparatus the lever CD seems to have been very massive as compared with JK, KH', or the bar itself. We shall suppose therefore the effect of the springs to be represented by a periodic force $Q = Q_0 \sin nt$ applied at D.

Let M be the mass of the lever CD, and k its swing radius about F; let $FC = c$, $FD = c'$, and let T be the whole force of the lever applied at C. Then if we suppose the weight of the lever to have a counterbalance at C, or to be just supported by the dynamometer in the initial position of R, we have for the periodic motion of the lever, if it be turned through an angle θ from this initial position,

$$Mk^2 \frac{d^2\theta}{dt^2} = Q_0 c' \sin nt - cT \quad \text{.....(i).}$$

Wöhler evidently *assumed that we might put* $M = 0$,

or
$$T = \frac{c'}{c} Q_0 \sin nt.$$

For the motion of the bar AB, we have

$$\Delta \frac{d^2u}{dt^2} = E \frac{d^2u}{dx^2} \quad \text{.....(ii),}$$

where u is the shift at distance x from B.

Now $u = 0$, when $x = 0$, and $E\alpha \dfrac{du}{dx} = T$, when $x = l$. Hence, if we neglect the effect of the initial speeds, u must be of the form

$$u = A \sin px \sin nt, \text{ where } p^2 = \frac{\Delta}{E} n^2.$$

Thus
$$T = E\alpha Ap \cos pl \sin nt.$$

Further
$$u = c\theta \text{ nearly;}$$

thus (i) becomes

$$\frac{Mk^2}{c^2} c \frac{d^2u}{dt^2} = Q_0 c' \sin nt - cE\alpha Ap \cos pl \sin nt,$$

or,
$$-M \frac{k^2}{c^2} cn^2 A \sin pl \sin nt = Q_0 c' \sin nt - cE\alpha Ap \cos pl \sin nt,$$

or,
$$A \left(Ep\alpha \cos pl - \frac{Mk^2}{c^2} n^2 \sin pl \right) = \frac{Q_0 c'}{c},$$

or,
$$A = \frac{Q_0 c'}{Ep c\alpha \left\{ \cos pl - \dfrac{M}{m} \dfrac{k^2}{c^2} pl \sin pl \right\}},$$

where
$$m = \text{the mass of the bar.}$$

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Now the maximum traction will be at the built-in end and be at such a time that

$$nt = (2i + 1)\tfrac{1}{2}\pi,$$

or it equals $\pm EAp$,

$$= \pm \frac{Q_0}{\alpha}\frac{c'}{c}\frac{1}{\cos pl - \dfrac{M}{m}\dfrac{k^2}{c^2}pl\sin pl}.$$

Now, if Q_0 had been applied as a statical load at D, we should have had the maximum traction

$$= \pm \frac{Q_0}{\alpha}\frac{c'}{c},$$

hence we find

$$\frac{\text{kinetic traction}}{\text{statical } \ldots\ldots} = \frac{1}{\cos pl - \dfrac{M}{m}\dfrac{k^2}{c^2}pl\sin pl};$$

or we must have

$$\frac{\text{kinetic safe load}}{\text{statical } \ldots\ldots} = \cos pl - \frac{M}{m}\frac{k^2}{c^2}pl\sin pl.$$

If pl be small, this may be written

$$= 1 - p^2l^2\left\{\tfrac{1}{2} + \frac{M}{m}\frac{k^2}{c^2}\right\}$$

$$= 1 - p^2l^2\frac{M}{m}\frac{k^2}{c^2} \text{ nearly.}$$

8. We can only get a rough approximation to the value of this expression, as we do not know the details of the mass, &c. of Wöhler's machine. Suppose, however, his lever CD to have multiplied the force at D, 100 times, and as it appears to have been very massive, let us take it 100 times the mass of the bar AB; then the ratio k^2/c^2 would have been about $\frac{100}{8}\frac{00}{}$. Further, suppose the bar to be of steel 1·5 feet long, then, if the force changes twice a second,

$$p^2l^2 = \frac{16\pi^2 \times (1{\cdot}5)^2}{16{,}000 \times 16{,}000} = \frac{9}{6{,}400{,}000} \text{ nearly,}$$

and

$$p^2l^2\frac{M}{m}\frac{k^2}{c^2} = \tfrac{3}{6}\tfrac{0}{4} = \tfrac{1}{2} \text{ nearly.}$$

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Hence we have

$$\frac{\text{kinetic safe load}}{\text{statical } \ldots\ldots} = \tfrac{1}{2} \text{ nearly.}$$

Or, the effect of the inertia of the lever may be of the utmost importance. I lay no particular stress on the above numbers, except as indicating that the inertia of Wöhler's apparatus may very possibly help to account for his results.*

9. Case (iii). *Continuous change from positive to negative traction without any inertia of loading.*

Suppose a bar of length l fixed at one end and the force $P' \cos pt$ applied at the other. Then, if α be the cross-section, and E the stretch-modulus as before, we have for longitudinal vibrations

$$\rho \frac{d^2\xi}{dt^2} = E \frac{d^2\xi}{dx^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where ξ is the shift of a point on the axis, distant x from the fixed end.

The terminal conditions are

$$E\alpha \frac{d\xi}{dx} = P' \cos pt, \quad \text{when } x = l \ldots\ldots\ldots \text{(ii)},$$

$$\xi = 0, \qquad \text{when } x = 0 \ldots\ldots\ldots\ldots\text{(iii)}.$$

Let $$\xi = A \sin mx \cos pt + \xi_0,$$

where the first term represents the particular integral chosen to remove the term in P' from (ii). We easily find, if

$$E/\rho = v^2,$$

$$p^2 = v^2 m^2,$$

$$AEm \cos ml = P'.$$

Whence the particular integral is

$$\xi_1 = \frac{P'v}{Eap \cos \dfrac{pl}{v}} \sin \frac{px}{v} \cos pt \quad \ldots\ldots\ldots\ldots\text{(iv)}$$

---

* If it seems that I have exaggerated the length and mass of Wöhler's lever, it must be remembered that his apparatus really consisted of *three* levers, and allowance ought to have been made for the inertia of *all* of them.

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The general integral is

$$\xi_0 = \Sigma A_q \sin\frac{qx}{v}\cos(qt+\beta).$$

Now suppose the motion started from the position in which the statical force P' produces uniform stretch in the bar, then initially

$$\frac{d\xi}{dt}=0, \quad \xi=\frac{P'}{Ea}x.$$

Hence $\beta=0$, $\frac{ql}{v}=\frac{1}{2}i\pi$, where i is an odd integer, and we have to determine A_q or A_i, the equation,

$$\frac{P'x}{Ea}-\frac{P'v}{Eap\cos\frac{pl}{v}}\sin\frac{px}{v}=\Sigma A_i \sin\frac{i\pi x}{2l}.$$

Or, by the usual Fourier's process,

$$A_i=\frac{2P'}{Eal}(-1)^i\frac{\frac{p^2}{v^2}\frac{16l^2}{i^4\pi^4}}{1-\frac{p^2}{v^2}\frac{4l^2}{i^2\pi^2}} \quad \text{(v)}.$$

Thus the value of ξ is fully determined. Let us put

$$\lambda = 2pl/(v\pi),$$

or

$$A_i=\frac{2P'}{Ea}(-1)^i\frac{4l}{i^4\pi^3}\frac{\lambda^2}{1-\lambda^2/i^2} \quad \text{(vi)}.$$

10. To obtain the traction at the fixed end, where it can be easily shown to reach its maximum value, we have

$$T_1=E\left(\frac{d\xi_1}{dx}+\frac{d\xi_0}{dx}\right), \text{ when } x=0,$$

$$=\frac{P'}{Ea}\left\{\frac{\cos pt}{\cos\frac{1}{2}\pi\lambda}+\Sigma(-1)^i\frac{4}{i^2\pi}\frac{\lambda^2}{1-\lambda^2/i^2}\cos\frac{i\pi v}{2l}t\right\} \quad \text{(vii)}.$$

Now, *if the change of stress be continued indefinitely, there may at last come a time in which all the important harmonics have the same sign, and their maximum amplitude at, or very nearly at, the same instant.* Hence the true limit to the maximum traction, which indeed may perhaps only be reached after a very long period, is

$$T_1=\frac{P'}{Ea}\left\{\frac{1}{\cos\frac{1}{2}\pi\lambda}+\Sigma\frac{4}{i^2\pi}\frac{\lambda^2}{1-\lambda^2/i^2}\right\}.$$

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Suppose P to be the safe statical load, then $T_1 = P/E\alpha$, and we find

$$\left.\begin{aligned} P' &= P/n \\ \text{where} \quad n &= \sec \tfrac{1}{2}\pi\lambda + \Sigma \frac{4}{i^2\pi} \frac{\lambda^2}{1-\lambda^2/i^2} \end{aligned}\right\} \quad \text{.....(viii).}$$

Thus we see that the limit to safe loading is not *theoretically* the same for the statical load and for the gradual change of stress. The *practical* importance of this result depends of course on how far the value of n for such cases as occur in practise differs from unity. I proceed to find its numerical value in one or two possible cases.

11. *Example* (*a*). Let us take a bar of wrought-iron, 10 feet long, to which the stress is applied 10 times a second, once a second and once in ten seconds.

Let us first calculate v, taking E=26,000,000 and its weight as 489 lbs. per cubic foot. It will be found to be almost exactly 15,800 feet per second. This is rather less than the experimental value (16,100 – 16,300), so that we may fairly take it in round numbers 16,000 feet per second. Hence, if $2\pi/p = m$ seconds, we have

$$\lambda = \frac{2pl}{v\pi} = \frac{4l}{mv} = \frac{1}{400m},$$

Obviously, if $m = 1$ or 10 this will be practically insensible. It is almost so, when $m = 1/10$, for then $\lambda = \frac{1}{40}$ and is still very small. We find very approximately

$$\begin{aligned} n &= \sec 2^\circ 15' + \frac{1}{400\pi}\{1 + \tfrac{1}{17} + \tfrac{1}{125}\} \\ &= 1{\cdot}0007716 + {\cdot}0008316 \\ &= 1{\cdot}0016032, \end{aligned}$$

or $$\begin{aligned} P' &= P/1{\cdot}0016, \text{ nearly,} \\ &= {\cdot}9984P, \end{aligned}$$

which is practically within the limits of experimental error.

Example (*b*). Suppose a long wire or cable of 40 feet* to be subjected to a gradual change of load 20 times a second, the velocity of sound in the cable being 16,000 feet per second. The suspending bars of a large bridge under the rapid motion

* Long chains in pits, &c. appear in some cases to lose tensile strength rapidly.

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of a train might possibly be subjected to something of this kind*, we have $\lambda = 1/5$, thus we find

$$n = \sec 18^\circ + \frac{4}{25\pi}\left\{\frac{1}{1-\frac{1}{25}} + \frac{1}{27}\,\frac{1}{1-\frac{1}{25.27}}\right\}$$

$$= 1{\cdot}0514622 + {\cdot}0530515 + {\cdot}0018863$$

$$= 1{\cdot}1064, \text{ nearly,}$$

or $$P' = {\cdot}90383P, \text{ nearly.}$$

Thus the reduction even in this extreme case is only about 1/10 of the safe statical load.

It is obvious that we should have to take very long cables or wires, or very quick repetitions of load such as hardly occur in practise, for the stress to accumulate owing to the vibratory action.

Hence, if small bars subjected for a very long time to a gradual change from positive to negative traction at a comparatively slow rate, say 10 times a second, ultimately break-down with half the statical safe loading, that fact cannot be explained by the kinetic accumulation of stress.

12. But it must be noted that the result just obtained supposes that:

(i) The load $P' \cos pt$ is due to some mechanism which *has itself no inertia.*

(ii) The bar at time $t = 0$ is under uniform stretch due to the traction P'.

Both of these suppositions are extremely unlikely to be true for the case of the mechanism by means of which experiments in repeated loading are made. The load is generally due to some system which itself possesses inertia (*i.e.* as in the case of the flexure experiments), and there is no absolute security that either in the frequent starting or stopping of the machinery, the bar may not suddenly be subjected to a load P' when it is at zero stretch throughout. In this latter case, we have at once for safety,

$$P' = \tfrac{1}{2}P.$$

---

* For example, taking the very unfavourable case of a train going 60 miles an hour, or 88 feet per second with the wheels four or five feet apart!

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This seems to me the probable explanation of the fact that repeated changes of load, *assumed* to be gradual, will ultimately destroy the bar, if

$$P' = \tfrac{1}{2}P.$$

For among those changes there are probably many which really amount to impulsive loading.

Thus I do not consider that the above apparently negative results for the case of traction demand that we should adopt a new theory of cohesion for bodies loaded below the safe statical limit, when they are subjected to gradually alternating stress. Far more exact experiments, in which all suspicion of the possible influence of the inertia of the load or of impulsive action has been removed, must be made before it would seem to me justifiable to seek for a new theory of elasticity to explain the peculiar phenomena of repeated loading.

University College, London,
March, 1890.

ON FINDING FACTORS.

By *Lt.-Col. Allan Cunningham, R.E.*, Fellow of King's Coll., Lond.

1. THE problem of resolving numbers into factors is one of great difficulty to effect in any reasonable compass in high numbers when the factors are very unequal; the worst case being that of a prime number whose only factors (being the number itself and one) have the maximum inequality.

In the following new process an attempt is made to reduce the excessive number of operations often required in some processes, whilst *each operation is of the simplest possible kind*, viz. either an addition or subtraction.

2. *Preliminary.* All even numbers being obviously divisible by 2, also all numbers ending in 5 or 0 being obviously divisible by 5 or 10; these factors (2, 5, 10) should be completely removed by actual division, and any other easily found prime factors, *e.g.* 3, 7, 11 should also—to shorten the process—be removed by actual division. The quotient can then only have one of the four odd numbers 1, 3, 7, 9 in its units place. Let this final quotient (which is the number to be operated on) be called N.

Extract the square root of N; or, which is simpler, look out N in a Table of Squares. If N turns out to be a perfect square, the factors have been found (each being $\sqrt{N}$).

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3. *Principle of process.* If N be not a perfect square, let the factors be X, Y, and write

$$X = A + B, \quad Y = A - B,$$

therefore $$N = XY = (A + B)(A - B) = A^2 - B^2,$$

so that the factors will be known if two perfect squares (A^2, B^2) can be found whose difference $= N$.

4. *Table of Squares.* The advantage of separation into the difference of squares consists essentially in replacing the labour of actual multiplication of trial factors X, Y by simple subtractions of square numbers A^2, B^2 which *can be found in a Table of Squares*, so that no actual multiplication, squaring, or extraction of roots should be required.

5. *Min. limit of A.* The equation $N = A^2 - B^2$ shows (since N is, by hypothesis, not a perfect square) that

$$\text{min. value of } A, \text{ say } A_1 = \text{integer next} > \sqrt{N}.$$

6. *Trial values of B.* Subtract N from *each* of the squares A_1^2, $(A_1 + 1)^2$, $(A_1 + 2)^2$, ..., $(A_1 + r)^2$ *in turn*, wherein r is any integer thought convenient. Compare each difference $(A_1 + r)^2 - N$ *in turn* with a Table of Squares.

If any of them be a perfect square, it is the sought square B^2, and the problem is solved.

If this process* be carried far enough, the value of B^2 will eventually be found: but the process is utterly too tedious when r is large (*i.e.* in fact when B is large), so a shorter process is required.

7. *Present process.* The present process depends on determining from the final digits of N what final digits are admissible in A^2, B^2. This is done digit by digit, rejecting all unsuitable digits as the process advances.

8. *Min. value of B.* Let this be B_1. By what precedes,

$$B_1^2 > (A_1^2 - N), \ldots, \text{(if this be not a perfect square)}.$$

Thus B_1 may be a small integer (its absolute minimum $= 1$), and further

$$B^2 > \{(A_1 + r)^2 - N\}, \text{ where } r = 1, 2, 3, \ldots, \textit{in turn} \text{ so long as these are not perfect squares.}$$

[N.B. If B_1 be a very small integer, a very few trials will rapidly raise this lower limit].

* This was in effect the process proposed in *Nature*, vol. 39, p. 413.

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9. *Max. limits of A, B.* Call these A_m, B_m. The equation $A^2 - B^2 = N$ shows that A_m, B_m increase together, and are therefore greatest together, viz. when their difference is least (or $= 1$), so that

$$\text{lowest factor } Y = A_m - B_m = 1,$$

$$\text{therefore} \qquad N = X = A_m + B_m,$$

$$\text{and} \qquad A_m = \tfrac{1}{2}(N+1),\ B_m = \tfrac{1}{2}(N-1),$$

and these are the only values when N is a prime.

10. *Practical maxima.* The power of the Table of Squares available absolutely limits these maxima to the highest number (say A_m) whose square is tabulated. In what follows it will be supposed that Barlow's Table of Squares is used, which gives squares to the limit of

$$A_m = 10{,}000.$$

11. *Max. number resolvible.* This depends chiefly on the lower factor, say Y, thus

$$A - B = Y,\ A + B = 2A - Y,\ N = (2A - Y)\,Y,$$

therefore

$$\text{highest number resolvible, } N = (2A_m - Y)\,Y.$$

This limit is low when Y is low, the lowest being when $Y = 1$, viz.

$$\left.\begin{matrix}\text{Highest prime recognisable}\\ \text{(without preliminary reduction)}\end{matrix}\right\} = 2A_m - 1,$$

and it increases with increase of Y. Hence it follows also that

If N be a number containing *none* of the *complete* series of prime factors 2, 3, 5, 7, &c., ..., z, then

$$\text{Highest number resolvible, } N_m = (2A_m - z)\,z.$$

which gives, when N is the number proposed,

$$\left.\begin{matrix}\text{Highest prime factor}\\ \text{to be excluded from } N\end{matrix}\right\} z = \text{integer next} < A_m - \sqrt{(A_m^2 - N)}.$$

12. *Preliminary Step.* The highest number N_m *obviously* solvible is therefore, since it may be a prime,

$$N_m = 2A_m - 1\ (= 19{,}999 \text{ with Barlow's Tables}).$$

The above, however, suggests a useful preliminary step for ascertaining whether a number N (exceeding that limit) falls within the powers of the Table of Squares available, and of reducing it in some cases within those powers, viz.

"Cast out by actual division *all* small prime factors 2, 3, 5, &c., ..., up to z, (where z is given by Art. 11)."

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13. *End digits of Squares.* An examination of a Table of Squares will show that in perfect squares there are only

6 possible units digits (0, 1, 4, 5, 6, 9).

22 possible groups of tens and units digits together.

159 possible groups of hundreds, tens and units.

1044 possible groups of the four final digits.

as shown in the Table* below. These will be called for shortness "possible end digits," or "possible endings."

POSSIBLE END DIGITS OF PERFECT SQUARES.

Thousands any of	Hundreds any of	Tens any of	Units any of
0 0, 2, 4, 6, 8 2 1, 3, 5, 7, 9	0 1, 4, 9 5 6	0	0
any digit any digit	0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 4, 8 2, 6	1
0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 3, 4, 7, 8 1, 2, 5, 6, 9	0, 8	4
0, 2, 4, 6, 8 1, 3, 5, 7, 9	2, 3, 6, 7 0, 1, 4, 5, 8, 9	2	
0, 2, 4, 6, 8 1, 3, 5, 7, 9	1, 2, 5, 6, 9 0, 3, 4, 7, 8	4	
0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 1, 4, 5, 8, 9 2, 3, 6, 7	6	
any digit 0, 5	0, 2 6	2	5
0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 1, 4, 5, 8, 9 2, 3, 6, 7	1, 9	6
0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 3, 4, 7, 8 1, 2, 5, 6, 9	3	
0, 2, 4, 6, 8 1, 3, 5, 7, 9	2, 3, 6, 7 0, 1, 4, 5, 8, 9	5	
0, 2, 4, 6, 8 1, 3, 5, 7, 9	1, 2, 5, 6, 9 0, 3, 4, 7, 8	7	
any digit any digit	0, 2, 4, 6, 8 1, 3, 5, 7, 9	0, 4, 8 2, 6	9

* A longer Table including all the possible 5th digits would be very useful for practical solution of factors, but it would be a much larger Table than can be here given, and would be beyond the power of Barlow's Tables to complete (except for squares ending in 0).

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14. *Equation of digits.* For any particular value of N the possible values of the end digits of A^2, B^2 are greatly limited by the necessity of satisfying the equation $N=A^2-B^2$ identically, *digit by digit*; thus giving rise to *one equation for every digit in* A^2.

Let n, a, b be the number of digits in N, A^2, B^2,
„ the symbol $\equiv$ denote arithmetical equality to the extent of the group of digits stated.

Then the equations are a in number, and are

(i) Diffce. of units digits of A^2, $B^2 \equiv$ units digits of N.

(ii) Diffce. of two end digits of A^2, $B^2 \equiv$ two end digits of N.

(iii) Diffce. of three end digits of A^2, $B^2 \equiv$ three end digits of N.

(iv) &c., &c., &c., $\equiv$ &c.,

the n^{th} equation being always

n^{th}. Diffce. of n end digits of A^2, $B^2 \equiv N$,

and this is the *last* equation when $a=n$; whilst, when $a>n$, each of the remaining $(a-n)$ equations is of form

$(n+r)^{\text{th}}$. Diffce. of $(n+r)$ end digits of A^2, $B^2=0$,

wherein r takes every integer value from 1 to $(a-n)$.

The groups of 1, 2, 3, ..., a digits formed as above will be styled "trial endings," and the digits themselves will be styled "trial digits."

15. *Trial values of* B^2, A^2. As B^2 may be a very small number (Art. 8) each of the above "trial endings" of B^2 which happens to be a *perfect square* should be looked on as a "trial value" of B^2 itself. Each such "trial value" of B^2 added to N gives a "trial value" of A^2 (since $A^2=B^2+N$). Any such "trial value" of A^2 which happens to be a perfect square is in fact the sought value of A^2, and the corresponding "trial value" of B^2 is the sought value of B^2: these therefore *complete the solution.*

16. *Curtailment of work.* To hasten the solution, the process of Art. 15 should be tried frequently, beginning of course from the time when the "trial endings" of B^2 exceed the highest of the "low limits" of B^2 found by Art. 8. These trials of Art. 15 are conveniently made either at end of each step or at end of each distinct part of a step of Art. 14.

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17. *Number of equations.* The number of "equations of digits" requiring to be *actually used* is now seen to be always b (the same as the number of digits in B^2) which varies with B^2; hence

b is very small when B is small, *i.e.* when X, Y are nearly equal.

b increases with B, *i.e.* as X, Y depart from equality.

b is a max. when B is a max., *i.e.* when X, Y have maximum inequality, in which case

$$Y = 1,\ X = N,\ B = \tfrac{1}{2}(N-1),\ b = (N-1).$$

Thus b may be very small, but is *liable to reach the high limit* of $(N-1)$ when N is a prime.

18. *Rate of work.* From Art. 18 it appears that the rate of work depends on the value of b; being quick when b is small, and tedious when b is large. This tediousness is caused not so much by the increase of the number of the equations themselves, as by the rapid increase (in a geometric ratio) of the *number of trial digits* at each step as b increases. In fact with high primes the process is so tedious as to be of no practical use.

[So long as b is not > 6, the work required is quite within practical limits, though laborious when $b > 5$].

19. *Example.* An example will make the process more clear. Let $N = 586{,}199$ (which is not a perfect square) to be solved by Barlow's Tables.

STEP I. As $N > 19{,}999$ (the ordinary limit of Barlow's Tables), apply the test of Art. 12. Here Art. 11 gives

$$z = \text{integer next} < \{10{,}000 - \sqrt{(10{,}000^2 - N)}\},\ \textit{i.e.}\ z = 30,$$

as the limit of small factors not within the power of the Tables. By actual trial division by all prime factors < 30, it will be found that none of them are factors of N; and by Art. 11

$$\text{Highest number resolvible} = (20{,}000 - 30) \times 30 = 591{,}100,$$

so that Barlow's Tables may be safely used.

STEP II. Barlow's Tables give (see Art. 5)

$$A_1 = (\text{integer next} > \sqrt{N}) = 766,$$

$$A_1^2 - N = 577,\ (A_1 + 1)^2 - N = 2090.$$

Hence, by Art. 8, $B^2 > 2090$ (because 577, 2090 are not perfect squares); so that the testing of the "trial values" of B^2 by Arts. 15, 16 may start from this value.

STEP III. *End digits of* A^2, B^2. The several parts of this Step are marked i, ii, iii, &c.

III—i. The Table of Art. 13 gives

$$\text{Possible units} \begin{cases} \text{of } A^2 = \text{any of } 0, 1, 4, 5, 6, 9 \\ \text{of } B^2 = \text{any of } 0, 1, 4, 5, 6, 9 \end{cases},$$

whilst by Eq. i., Diffce. of units of A^2, $B^2 \equiv 9$.

therefore

$$\text{Trial units (only 4 pairs)} \begin{cases} \text{of } A^2 \text{ are } 0, 4, 5, 9 \\ \text{of } B^2 \text{ are } 1, 5, 6, 0 \end{cases},$$

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III—ii. Again, accepting these "trial units," the same Table gives the following system of

Possible tens and units.

Only 00 for A^2, with any of 01, 21, 41, 61, 81 for B^2.

Any of 04, 24, 44, 64, 84 for A^2, with 25 for B^2.

Only 25 for A^2, with any of 16, 36, 56, 76, 96 for B^2.

Any of 09, 29, 49, 69, 89 for A^2, with 00 for B^2,

whilst by Eq. ii., Diffce. of tens and units of A^2, $B^2 \equiv 99$.

therefore Trial tens and units (only 2 pairs) $\left\{\begin{matrix}\text{of } A^2 \text{ are } 00, 24\\ \text{of } B^2 \text{ are } 01, 25\end{matrix}\right\}$.

III—iii. Again, accepting these 2 trials tens and units, the same Table gives the system of

Possible groups of three end digits.

A^2, any of 000, 100, 400, 500, 600, 900
B^2, any of 001, 201, 401, 601, 801 }

A^2, any of 024, 124, 224, 324, 424, 524, 624, 724, 824, 924
B^2, any of 025, 225, 625 }

whilst by Eq. iii., Diffce. of three end digits of A^2, $B^2 \equiv 199$.

therefore Trial last 3 digits (only 6 pairs) $\left\{\begin{matrix}\text{of } A^2 \text{ are } 000, 400, 600, 224, 424, 824\\ \text{of } B^2 \text{ are } 801, 201, 401, 025, 225, 625\end{matrix}\right\}$.

III—iv. Again, accepting these 6 "trial end triplets," the same Table gives the following system of possible thousands digits

Possible thousands digit of A^2		End triplet of A^2	Possible thousands digit of B^2		End triplet of B^2
only 0	with	000	any digit	with	801
any of 0, 2, 4, 6, 8	with	400	any digit	with	201
any of 1, 3, 5, 7, 9	with	600	any digit	with	401
any of 0, 2, 4, 6, 8	with	224	any digit	with	025
any of 1, 3, 5, 7, 9	with	424	any digit	with	225
any of 1, 3, 5, 7, 9	with	824	either 0, 5	with	625

whilst by Eq. iv., Diffce. of last 4 digits of A^2, $B^2 \equiv 6199$.

This gives the following system of "trial end quartetts" of A^2, B^2 (only 22 pairs in all),

A^2	B^2	A^2	B^2	A^2	B^2	A^2	B^2
0000	3801	.....	.....	.....	.....	.....	.....
0400	4201	1600	5401	0224	4025	1424	5225
2400	6201	3600	7401	2224	6025	3424	*7225
4400	8201	5600	9401	4224	8025	5424	9225
6400	0201	7600	1401	6224	0025	7424	*1225
8400	2201	9600	3401	8224	*2025	9424	3225
.....	.....	.....	.....	.....	.....	1824	5625

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As the lower limit of B^2 (2090, see Step II) has now been exceeded, the "trial end quartetts" of B^2 should be looked out in the Table of Squares. The three marked* are the only perfect squares: out of these the value 7225 (the only one > 2090) is therefore (by Art. 15, 16) a "trial value" of B^2 itself. But on adding it to N, the "trial value" of A^2 (= $B^2 + N$, see Art. 15) resulting turns out to be not a perfect square, so that these trial values are rejected.

III—v. Again, accepting the above 22 "trial end quartetts, the possible 5th digits may be taken from the Table of Art. 13 for those values of A^2 which end in 00: but for all the rest, reference must now be made to the Table† of Squares itself (as the Table of Art. 13 extends only to 4 digits). A large Table of Squares gives the system of possible 5th digits below, which is continued‡ only as far as those values of A^2, B^2 which prove on actual trial (as shown below) to be the sought values.

Possible 5th digit of A^2	End quartett of A^2	Possible 5th digit of B^2	End quartett of B^2
any of 0, 1, 4, 5, 6, 9	0000	any of 0, 1, 2, 3, (4, 5, 6, 7, 8), 9	3801
any digit	0400	any of 0, 1, 2, 3, (4, 5, 6, 7, 8), 9	4201
any digit	2400	any of 0, 1, 2, 3, (4, 5, 6, 7, 8), 9	6201
any digit	4400	any of 0, 1, 2, 3, (4, 5, 6, 7, 8), 9	8201
any digit	6400	any of 0, 1, 2, 3, 4, (5, 6, 7, 8), 9	0201
any digit	8400	any of 0, 1, 2, 3, 4, (5, 6, 7, 8), 9	2201

N.B. The values of 5th digit of B^2 enclosed in brackets (4, 5, 6, 7, 8) are not contained in Barlow's Tables, so are not really required for this actual case,

whilst by Eq. v., Diffce. of last 5 digits of A^2, $B^2 \equiv 86199$.

This gives the following system of "trial and quintetts" of A^2, B^2; excluding all cases not in Barlow's Tables.

	Trial Fifth Digits from Eq. v.	Trial End Quartetts	Perfect Squares of B^2
A^2	0, 1, ...	0000	.....
B^2	1, 2, ...	3801	none
A^2	0, 1, 2, ... 8, 9	0400	.....
B^2	1, 2, 3, ... 9, 0	4201	none
A^2	0, 1, 2, ... 8, 9	2400	.....
B^2	1, 2, 3, ... 9, 0	6201	none
A^2	0, 1, 2, ... 8, 9	4400	.....
B^2	1, 2, 3. ... ... 9, 0	8201	none
A^2	0, 1, 2, 3, ... 8, 9	6400	.....
B^2	1, 2, 3, 4, ... 9, 0	0201	10,201
A^2	0, 1, 2, 3, ... 8, 9	8400	.....
B^2	1, 2, 3, 4, ... 9, 0	2201	22,201

* These numbers are the only perfect squares.

† It is comparatively easy to pick these out from Barlow's Tables with a little practice. The Tables contain 50 squares in a page, or 100 on 2 opposite pages: this arrangement greatly helps the picking out the required figures, as they then fall on similar parts of each page.

‡ It is quite unnecessary to continue the Table, because the caution of Art. 16 causes the solution to be discovered by the time the Table has been extended as far as printed.

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The only perfect squares in the above "trial end quintetts" of B^2 are 10,201, 22,201, which are therefore the only trial values of B^2 in the Table (see Art. 15). Adding these successively to N the second alone gives rise to a perfect square, viz.

$$B^2 + N = 22{,}201 + 586{,}199 = 608{,}400 = A^2,$$

so that the sought values of A^2, B^2 are

$$A^2 = 780^2,\ B^2 = 149^2,$$

whence the sought factors are

$$X = A + B = 929,\ Y = A - B = 631.$$

Remarks. It will be seen that the shortness of the work in this example depends chiefly on the small number of digits in B^2, viz. $\delta = 5$; and partly on the small value of $B^2 = 22{,}201$ requiring the Tables in Step III—v. to be extended to only 6 lines for the "possible digits" and to 12 lines for the "trial digits;" whereas these Tables would have run to 22 and 44 lines respectively, if it had been necessary to work them out completely. Had δ been > 5 (say $= 6$) the next step to III—vi. would have run to much greater length (see Art. 18).

SUR UNE PARABOLE LIÉE À UNE CONIQUE PAR CERTAINES PROPRIÉTÉS REMARQUABLES.

Par M. le Professeur *Mannheim*.

CHASLES, à la page 145 de son traité des sections coniques, démontre le théorême suivant:

1. *Si, autour d'un point pris dans le plan d'une conique, on fait tourner une transversale et que par son pôle on mène une perpendiculaire à cette droite: ces perpendiculaires successives enveloppent une parabole qui est tangente à la polaire du point fixe et aux tangentes à la conique menées par les pieds de ses normales abaissées de ce point.*

Je vais reprendre ce théorême, en donner une nouvelle démonstration et faire connaître quelques résultats qui n'ont pas été remarqués. Je retrouverai aussi des propriétés connues. Appelons C la conique donnée, dont le centre est o, d le point fixe pris dans son plan et D la polaire de ce point par rapport à C. Une transversale quelconque, issue de d, a son pôle sur D. On peut alors énoncer le théorême précédent sous cette forme bien connue:

2. *Les perpendiculaires abaissées respectivement des points d'une droite D sur les polaires de ces points enveloppent une parabole etc.*

Soit m un point de D. Par ce point faisons passer une conique concentrique et homothétique à C. La tangente en m à cette conique est conjuguée de om, c'est-à-dire parallèle à la polaire de m par rapport à C.

La normale en m à la conique homothétique n'est autre alors que la perpendiculaire abaissée de ce point sur sa polaire

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par rapport à C. Les segments comptés sur cette normale, et compris entre m et les points où cette droite rencontre les axes de C, sont dans le rapport des carrés des axes de la conique qui passe par m, c'est à dire dans le rapport des carrés des axes de C puisque les deux coniques sont concentriques et homothétiques.

Quel que soit le point pris sur D, on a alors une droite menée de ce point, qui est partagée par les axes de C et par D en segments proportionnels. Ces droites enveloppent alors une parabole tangente aux droites qui servent à l'obtenir: les axes de C et la droite D.

Appelons P cette parabole. Je vais montrer qu'elle est encore tangente à un certain nombre d'autres droites.

Pour trouver ces droites, il suffit de prendre m dans différentes positions remarquables. Lorsque m vient en a ou b, point de rencontre de D et de C, les droites correspondantes sont les normales en ces points à C, ainsi:

3. *P est tangente aux normales à C dont les pieds sont les points de rencontre de cette courbe et de D.*

Lorsque m vient en i, milieu de ab, comme la polaire de i est la parallèle à D menée du point d, la perpendiculaire abaissée de i sur cette polaire est perpendiculaire à D, ainsi:

4. *P est tangente à la perpendiculaire à D élevée du milieu i de ab.*

Lorsque m vient au point de rencontre de D avec l'une des directrices de C, la perpendiculaire abaissée de ce point sur sa polaire passe par le foyer, correspondant à la directrice, ainsi:

5. *P est tangente aux perpendiculaires élevées respectivement des foyers f, f' de C aux droites $fd, f'd$.*

Enfin, il faut noter encore comme Chasles l'a remarqué que:

6. *P est tangente aux droites qui touchent C aux pieds des normales à cette courbes abaissées du point d.*

Ou, *les normales à C, abaisseés de d, ont pour pieds les points de contact des tangentes communes à cette courbe et à P.*

Faisons maintenant quelques remarques. P étant déterminée par les axes de C, la droite D et le rapport des carrés des axes de C, on peut remplacer cette courbe par une conique qui lui est concentrique et homothétique. Il résulte de là que:

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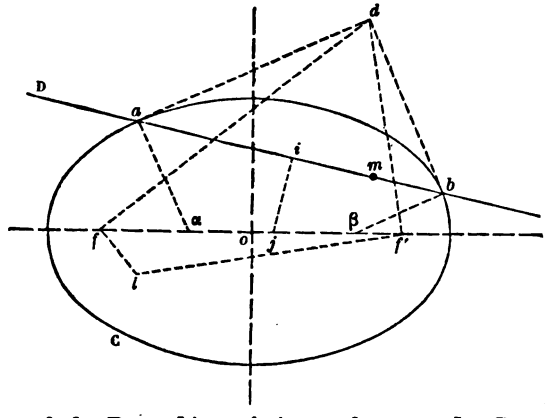
7. *Des points où une droite D est coupée par des coniques concentriques et homothétiques à C on élève respectivement des normales à ces courbes: toutes ces droites sont tangentes à une parabole etc.*

Parmi toutes ces coniques, celle qui est tangente à D touche cette droite au point i. On a alors en i pour cette conique particulière C_i deux normales confondues. La parabole P touche alors cette normale au centre de courbure de C_i. On retrouve ainsi cette propriété.

8. *La parabole tangente aux axes d'une conique, qui touche aussi la tangente et la normale en i à cette courbe, est tangente à cette normale au centre de courbure de la conique pour le point i.*



La parabole P est déterminée par les axes de C et par les perpendiculaires fl, $f'l$ menées des foyers f et f' aux droites fd, $f'd$. Si l'on remplace C par une conique homofocale et que l'on construise, relativement à cette courbe, l'enveloppe des perpendiculaires aux transversales menées de d qui sont abaissées des pôles de ces droites, on doit retrouver P. Comme cette parabole est tangente à la polaire de d, il résulte de là que.

9. *Les polaires d'un point d, par rapport à des coniques homofocales de foyers f, f' enveloppent une parabole tangente aux axes des coniques et tangente aux bissectrices des angles formés par fd et f'd.*

10. *Les pieds des normales à ces coniques homofocales qui sont issues de d appartiennent alors à une podaire de parabole.*

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*Cette podaire est évidemment aussi les lieu des points de contact des tangentes issues de d et le lieu des pieds des perpendiculaires abaissées de d sur les polaires de ce point par rapport aux coniques homofocales.**

On sait que toutes les tangentes à une parabole divisent deux tangentes à cette courbe en segments proportionnels.

De là diverses conséquences relatives aux tangentes de *P*. Appelons α et β les points de rencontre de l'axe focal ff' et des normales en a et b à C, et j le point où la perpendiculaire à D élevée du point i rencontre ce même axe.

On voit que:

11. *La perpendiculaire ij à D partage en deux parties égales le segment* $\alpha\beta$, ou encore:

Les projections de $a\alpha$ *et de* $b\beta$ *sur D sont égales.*

Sous cette forme, on retrouve un théorême connu dû à Laguerre. Mais notre théorême ne s'applique pas seulement aux projections des segments des normales en a et b compris entre ces points et l'un des axes de C, il s'applique aussi:

12. *Aux projections sur D des normales en a et b limitées à l'une quelconque des tangentes à P.*

On peut dire également:

13. *La tangente, qui touche C au pied d'une normale à cette courbe abaisseé de d, est partagée en segments égaux par* $a\alpha$, ij, $b\beta$.

14. *Cette même tangente est partagée en segments égaux par* lf, lf' *et par l'axe perpendiculaire à* ff'.

Si revenant au théorême 8 l'on applique la propriété des tangentes à une parabole, de déterminer des segments proportionnels sur deux tangentes fixes, on arrive à un grand nombre de constructions du centre de courbure d'une conique.†

---

* Nous retrouvons ainsi des propriétés bien connues.

† Voir mon *Cours de Géométrie descriptive*, 2e édition, page 173.

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Prof. Cayley, "Note on Schläfli's modular equation for the cubic transformation."

Mr. H. W. Segar, "Some inequalities (continued)."

Articles for insertion will be received by the Editor, or by Messrs. Metcalfe and Co. Limited, Printing Office, Trinity Street, Cambridge.

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ON THE SURFACES WHOSE LINES OF CURVATURE ARE ALL PLANE.

By *W. Burnside, M.A.*

If a correspondence is established, according to the method used by Gauss, between points on a surface and points on a sphere, so that P on the surface and P' on the sphere are corresponding points when the normals at P and P' are parallel; then it is well known that the tangents at corresponding points on a line of curvature on the surface and on its spherical representative are parallel. It follows that if the surface have a plane line of curvature its spherical representative is a circle, and that if the lines of curvature of both systems on the surface are plane, they must correspond to two sets of orthogonal circles on the sphere.

M. Bertrand (*Calcul Différentiel*, §§ 705, 706) uses this method to obtain the differential equation of the surfaces all whose lines of curvature are plane.

The analysis is very much simplified and the differential equation is presented in a form more immediately integrable by using a method due to M. O. Bonnet, which is explained and largely applied by M. Darboux in his *Leçons sur la theorie générale des surfaces* (*Première Partie*, §§ 163, 164, et seq.). The method consists in using as independent variables the parameters of the lines which correspond to the imaginary generators (or line of zero length) on the sphere.

If $$lx + my + nz + p = 0$$

be the equation to any tangent plane to the surface (l, m, n being direction-cosines), then we may write

$$l = \frac{\beta + \alpha}{\beta\alpha + 1}, \quad m = i\frac{\beta - \alpha}{\beta\alpha + 1}, \quad n = \frac{\beta\alpha - 1}{\beta\alpha + 1};$$

giving $$l^2 + m^2 + n^2 = 1,$$

$$dl = \frac{(1-\alpha^2)\,d\beta + (1-\beta^2)\,d\alpha}{(\beta\alpha+1)^2}, \quad dm = i\frac{(1+\alpha^2)\,d\beta - (1+\beta^2)\,d\alpha}{(\beta\alpha+1)^2},$$

$$dn = 2\frac{\alpha\,d\beta + \beta\,d\alpha}{(\beta\alpha+1)^2},$$

and $$dl^2 + dm^2 + dn^2 = \frac{4\,d\alpha\,d\beta}{(\beta\alpha+1)^2}.$$

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If now we put

$$\varpi = p\,(\beta\alpha + 1),$$

the equation to the tangent plane is

$$(\beta + \alpha)\,x + i\,(\beta - \alpha)\,y + (\beta\alpha - 1)\,z + \varpi = 0 \ldots\ldots \text{(i)},$$

and the tangential equation to the surface will be obtained when ϖ is known as a function of α and β.

The values of α and β corresponding to real points on the surface are conjugate imaginaries.

The coordinates of the point of contact of the tangent plane are obtained by combining (i) with the results of differentiating it with respect to α and β, viz.

$$x - iy + \beta z + p = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)},$$

$$x + iy + \alpha z + q = 0 \ldots\ldots\ldots\ldots\ldots\ldots \text{(iii)},$$

where $p = \partial\varpi/\partial\alpha, \quad q = \partial\varpi/\partial\beta.$

These give for a consecutive point

$$dx - i\,dy + \beta\,dz + dp + z\,d\beta = 0,$$

$$dx + i\,dy + \alpha\,dz + dq + z\,d\alpha = 0.$$

By the previous geometrical statement, if dx, dy, dz is an element of a line of curvature,

$$dx : dy : dz = dl : dm : dn.$$

Now

$$dl - i\,dm + \beta\,dn = \frac{2}{\beta\alpha + 1}\,d\beta,$$

$$dl + i\,dm + \alpha\,dn = \frac{2}{\beta\alpha + 1}\,d\alpha,$$

and hence along a line of curvature

$$dp + \left(z - \frac{2R}{\beta\alpha + 1}\right) d\beta = 0,$$

$$dq + \left(z - \frac{2R}{\beta\alpha + 1}\right) d\alpha = 0,$$

where R is some unknown function of α and β (in fact the corresponding principal radius of curvature).

Hence the differential equation of the lines of curvature is

$$dp\,d\alpha - dq\,d\beta = 0,$$

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or $$r\,d\alpha^2 - t\,d\beta^2 = 0 \quad \text{.....(iv)},$$

where $$r = \partial^2\varpi/\partial\alpha^2, \quad t = \partial^2\varpi/\partial\beta^2.$$

It is well known that the only sets of orthogonal circles on a sphere are those in which the sphere is cut by two sets of planes passing respectively through two lines conjugate with respect to it.

By properly choosing the axes the differential equations of the orthogonal systems of circles may be thrown into one of the three forms

$$\beta^2 d\alpha^2 - \alpha^2 d\beta^2 = 0 \quad \text{.....(v)},$$

$$d\alpha^2 - d\beta^2 = 0 \quad \text{.....(vi)},$$

$$(\beta^2 + \kappa^2)^2\, d\alpha^2 - (\alpha^2 + \kappa^2)^2\, d\beta^2 = 0 \quad \text{.....(vii)}.$$

Of these, (v) corresponds to the case where one set of planes passes through a diameter and the other set is perpendicular to it: (vi) to the case where the two sets pass respectively through two tangent lines at right angles at a point on the sphere; while in (vii) κ^2-1/κ^2+1 and κ^2+1/κ^2-1 are the distances from the centre of the two conjugate lines through which the planes pass.

The third form contains the other two as particular cases, viz. $\kappa = \pm 1$ and $\kappa = 0$ or ∞, but not in the most suitable form for application.

The differential equation corresponding to (v),

$$\alpha^2 \frac{\partial^2\varpi}{\partial\alpha^2} - \beta^2 \frac{\partial^2\varpi}{\partial\beta^2} = 0,$$

gives $$\varpi = f(\alpha\beta) + \sqrt{(\alpha\beta)}\, F\left(\frac{\alpha}{\beta}\right) \quad \text{.....(v')},$$

and that corresponding to (vi) gives

$$\varpi = f(\alpha + \beta) + F(\alpha - \beta) \quad \text{.....(vi')}.$$

The differential equation for the general form

$$(\alpha^2 + \kappa^2)^2 \frac{\partial^2\varpi}{\partial\alpha^2} - (\beta^2 + \kappa^2)^2 \frac{\partial^2\varpi}{\partial\beta^2} = 0,$$

on substituting

$$\alpha' = \tan^{-1}\alpha/\kappa, \qquad \beta' = \tan^{-1}\beta/\kappa,$$

$$\varpi' = \varpi \cos\alpha' \cos\beta',$$

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becomes $$\frac{\partial^2 \varpi'}{\partial \alpha'^2} - \frac{\partial^2 \varpi'}{\partial \beta'^2} = 0;$$

and gives

$$\varpi = (\alpha+\beta) f\left(\frac{\kappa^2 - \alpha\beta}{\alpha+\beta}\right) + (\alpha-\beta) F\left(\frac{\kappa^2+\alpha\beta}{\alpha-\beta}\right) \ldots \text{(vii')}.$$

Hence, writing n_0 for $\kappa^2 - 1/\kappa^2 + 1$, and remembering that $\varpi = p(\beta\alpha+1)$, the tangent plane to the surface may be written

$$lx + my + nz + lf\left(\frac{n-n_0}{l}\right) + mF\left(\frac{1-nn_0}{m}\right) = 0,$$

and every surface whose lines of curvature are all plane may be generated as the envelope of this plane.

The point equations are obtained at once from (i), (ii), and (iii).

These give

$$z = \frac{\varpi - p\alpha - q\beta}{\beta\alpha + 1},$$

$$-2x = (\alpha+\beta) z + p + q,$$

$$-2iy = (\alpha-\beta) z - p + q.$$

The special form (v') contains the surfaces of revolution, for if $F=0$,

$$\varpi = f(\alpha\beta),$$

which is the same thing as

$$p = f(n).$$

Now, if $f=0$,

$$\varpi = \sqrt{(\alpha\beta)}\, F\left(\frac{\alpha}{\beta}\right),$$

and to interpret this, writing $l = \sin\theta\cos\phi$, &c.,

$$p = \sin\theta F(\phi),$$

which represents any curve whatever in the plane of xy.

Hence the surfaces represented by

$$\varpi = f(\alpha\beta) + \sqrt{(\alpha\beta)}\, F\left(\frac{\alpha}{\beta}\right)$$

are such that their sections by planes parallel to xy (lines of curvature) when projected orthogonally on the plane of xy, form a system of parallel curves; while the common normals to these parallel curves are the orthogonal projection of the lines of curvature of the other system.

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The most general point-equations of such surfaces may, from geometrical considerations, be written

$$x = f(u)\cos v + F(v)\cos v - F'(v)\sin v,$$
$$y = f(u)\sin v + F(v)\sin v + F'(v)\cos v,$$
$$z = u.$$

The simplest surfaces with lines of curvature all plane, other than surfaces of revolution, will clearly be given by (vi′). When f and F are linear functions of their variables, the result is a sphere. As an example, put

$$\varpi = c - \tfrac{1}{4}a(\alpha+\beta)^2 + \tfrac{1}{4}b(\alpha-\beta)^2,$$

so that
$$p = -\tfrac{1}{2}a(\alpha+\beta) + \tfrac{1}{2}b(\alpha-\beta),$$
$$q = -\tfrac{1}{2}a(\alpha+\beta) - \tfrac{1}{2}b(\alpha-\beta),$$

$$z = \frac{\varpi - p\alpha - q\beta}{\beta\alpha + 1} = \frac{c + \tfrac{1}{4}a(\alpha+\beta)^2 - \tfrac{1}{4}b(\alpha-\beta)^2}{\beta\alpha+1},$$

$$-2x = (\alpha+\beta)(z-a), \quad -2iy = (\alpha-\beta)(z-b),$$

and
$$z\left(1 + \frac{x^2}{(z-a)^2} + \frac{y^2}{(z-b)^2}\right) = c + \frac{ax^2}{(z-a)^2} + \frac{by^2}{(z-b)^2},$$

or
$$(z-a)(z-b)(z-c) + (z-b)x^2 + (z-a)y^2 = 0.$$

If $\alpha = u + iv$, $\beta = u - iv$, then, putting $c = 0$,

$$x = \frac{u\{a + (a-b)v^2\}}{1+u^2+v^2}, \quad y = \frac{v\{b + (b-a)u^2\}}{1+u^2+v^2},$$

$$z = \frac{au^2 + bv^2}{1+u^2+v^2},$$

and u, v are the parameters of the lines of curvature. The lines of curvature are all ellipses, and all those of either system pass through a pair of common points, which will be real for one of the systems if a and b have the same sign.

It may be shewn without much difficulty that all surfaces of the third degree whose lines of curvature are plane in both systems are included in the equation just obtained.

The linear element of this surface is given by

$$ds^2 = \frac{[a + (a-b)v^2]^2\,du^2 + [b + (b-a)u^2]^2\,dv^2}{(1+u^2+v^2)^2},$$

so that the lines of curvature have the property of cutting up the surface into squares, or, in other words, they form a system of isothermal curves on the surface.

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As a second example, put

$$\varpi = \tfrac{3}{4}\left[(\alpha+\beta)^3 + (\alpha-\beta)^3\right] + \tfrac{1}{16}\left[(\alpha+\beta)^4 - (\alpha-\beta)^4\right]$$
$$= \tfrac{1}{2}(\alpha^2+\beta^2)(3+\alpha\beta),$$

which is the simplest form of ϖ leading to integral expressions for x, y, and z.

The resulting expressions are

$$x = 3u + 3uv^2 - u^3,$$
$$y = 3v + 3vu^2 - v^3,$$
$$z = 3v^2 - 3u^2,$$

where again u and v are the parameters of the lines of curvature. This is the minimum surface of M. Enneper (Darboux, *loc. cit.*, § 207). The form quoted above for a principal radius of curvature leads directly to the equation

$$z + \frac{\partial^2 \varpi}{\partial\alpha\partial\beta} = 0,$$

as the condition that the principal radii of curvature may be equal and opposite; and this condition is satisfied identically by the above values of z and ϖ.

The question treated in this note has been solved in many different ways, of which M. Darboux in the above-mentioned work gives one and refers to a second (*loc. cit.*, § 102).

SOME INEQUALITIES.

By *Hugh W. Segar*, Trinity College, Cambridge.

I. Our object is to generalize the result of § 1 in the last paper of vol. XIX. of this journal. This result is:—

$$\left.\begin{array}{ll} \text{if} & m > n, \\ \text{then} & \Sigma(a^m b^n - a^n b^m) \gtrless 0 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(\alpha),$$

according as a, b, c belong to the first or second of the systems

$$\left.\begin{array}{l} a>b>c \\ b>c>a \\ c>a>b \end{array}\right\}, \qquad \left.\begin{array}{l} a<b<c \\ b<c<a \\ c<a<b \end{array}\right\};$$

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The result (α) may be put thus

$$\begin{vmatrix} a^\alpha, & a^\beta, & a^\gamma \\ b^\alpha, & b^\beta, & b^\gamma \\ c^\alpha, & c^\beta, & c^\gamma \end{vmatrix} \gtrless 0 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where $$\alpha > \beta > \gamma.$$

We shall show a similar result to hold for any number of letters.

All letters are used to denote positive quantities only.

II. It is first necessary to establish the theorem:

"Any $n+1$ homogeneous functions of n letters are connected by a linear relation, which is such that if the functions be arranged in the order of their degrees, the terms are alternately positive and negative."

Consider three letters a, b, c.

The inequality (1) necessitates

$$\begin{vmatrix} H_{m+2}, & H_{n+2}, & H_2 \\ H_{m+1}, & H_{n+1}, & H_1 \\ H_m, & H_n, & H_0 \end{vmatrix} < 0 \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

whenever $m > n$. We thus get an infinite number of determinants, all negative, and such that any one can be transformed into any other multiplied by some function of a, b, c, by means of the relations which connect any four homogeneous functions. This function of a, b, c must be positive, and this requires the relation to be of the form

$$(H_\alpha) - (H_\beta) + (H_\gamma) - (H_\delta) = 0 \ldots\ldots\ldots\ldots\ldots (3),$$

where $$\alpha > \beta > \gamma > \delta,$$

and (H_n) denotes the product of H_n and some positive quantity.

Let now h refer to b, c, d, and H to a, b, c, d.

We have, say,

$$h_n - p_1 h_{n-1} + p_2 h_{n-2} - p_3 h_{n-3} = 0.$$

Now $$H_n = a^n + a^{n-1} h_1 + a^{n-2} h_2 + \ldots + h_n \ldots\ldots\ldots (\beta).$$

Hence $$H_n - p_1 H_{n-1} + p_2 H_{n-2} - p_3 H_{n-3} = a^n;$$

therefore $$H_3 - (H_2) + (H_1) - (H_0) = a^3.$$

Also $$H_4 - (H_3) + (H_2) - (H_1) = a^4.$$

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Eliminating H_1, we get

$$(H_4) - (H_3) + (H_2) - (H_0) = (x) \ldots\ldots\ldots (4),$$

where (x) stands simply for some positive quantity. For, the process of deriving this relation is the same as that for deriving

$$(h_4) - (h_3) + (h_2) - (h_1) = 0,$$

and we know that in this the coefficients of h_4, h_3, h_2, h_0 are alternately positive and negative.

Similarly $(H_5) - (H_4) + (H_3) - (H_2) = (x)$.

Eliminating H_2 by means of (4), we get

$$(H_5) - (H_4) + (H_3) - (H_0) = (x) \ldots\ldots\ldots (5).$$

Thus we have generally

$$(H_{n+2}) - (H_{n+1}) + (H_n) - (H_0) = (x).$$

Take $(H_6) - (H_5) + (H_4) - (H_0) = (x)$.

This, with (5), gives

$$(H_6) - (H_5) + (H_3) - (H_0) = (x).$$

In like manner

$$(H_7) - (H_6) + (H_3) - (H_0) = (x),$$

and so on.

This suffices to shew that, assuming the proposition for three letters, it follows for four, then in like manner for five, and so is generally true.

III. Consequently

$$\begin{vmatrix} H_0, & H_\alpha, & H_\beta, & \ldots \\ H_1, & H_{\alpha+1}, & H_{\beta+1}, & \ldots \\ H_2, & H_{\alpha+2}, & H_{\beta+2}, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \end{vmatrix}$$

is of same sign as

$$\begin{vmatrix} H_{2n-2}, & H_{2n-3}, & \ldots, & \ldots \\ H_{2n-3}, & H_{2n-4}, & \ldots, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & H_0 & & \end{vmatrix} \ldots\ldots\ldots\ldots (6),$$

where H refers to n letters a, b, c, ..., and the successive suffixes in rows and columns decrease by unity.

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Let h refer to $n-1$ letters $b, c, d, \ldots$. Then, by means of (β) and

$$H_n - aH_{n-1} = h_n,$$

we can reduce (6) to another determinant in terms of $h_0, h_1, h_2, \ldots$, such that only the last row contains a and its powers. Expanding in powers of a, all the coefficients except that of a^n consist of determinants of n rows and columns, and these in virtue of the relation between n homogeneous functions of $b, c, d, \ldots$ all vanish.

Thus (6) is of same sign as

$$(-1)^{n-1} \begin{vmatrix} h_{2n-3}, & h_{2n-4}, & \ldots, & \ldots \\ h_{2n-4}, & h_{2n-5}, & \ldots, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & & h_1 \end{vmatrix}.$$

We know that for three letters this determinant is negative. Hence for n letters it is positive or negative according as $(-1)^{\frac{1}{2}\{n(n-1)\}}$ is positive or negative. But

$$\begin{vmatrix} a^\alpha, & a^\beta, & a^\gamma, & \ldots \\ b^\alpha, & b^\beta, & b^\gamma, & \ldots \\ c^\alpha, & c^\beta, & c^\gamma, & \ldots \\ \ldots\ldots\ldots\ldots\ldots & & & \end{vmatrix}$$

$$= (-1)^{\frac{1}{2}\{n(n-1)\}} \begin{vmatrix} H_\alpha, & H_\beta, & H_\gamma, & \ldots \\ H_{\alpha-1}, & H_{\beta-1}, & H_{\gamma-1}, & \ldots \\ H_{\alpha-2}, & H_{\beta-2}, & H_{\gamma-2}, & \ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots & & & \end{vmatrix} \zeta^{\frac{1}{2}}(a, b, c, \ldots).$$

Hence, when $\alpha > \beta > \gamma \ldots$, the sinister is of the same sign as

$$\zeta^{\frac{1}{2}}(a, b, c, \ldots),$$

i.e., if the columns be arranged in descending order of magnitude of their degrees, the determinant is positive or negative with

$$(a-b)(a-c)(a-d)\ldots$$
$$(b-c)(b-d)\ldots$$
$$(c-d)\ldots$$
$$\&c.$$

Hence, generally, whatever be the orders of magnitude of $a, b, c, \ldots$, and of $\alpha, \beta, \gamma, \ldots$, our determinant is positive or negative with $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)\,\zeta^{\frac{1}{2}}(a, b, c, \ldots)$.

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IV. If we have

$$p_0 H_n - p_1 H_{n-1} + p_2 H_{n-2} - \ldots (-1)^n H_0 = 0,$$

we shewed in last paper

$$\frac{H_0}{H_1} < \frac{H_1}{H_2} < \frac{H_2}{H_3} < \ldots,$$

and

$$\frac{p_0}{p_1} < \frac{p_1}{p_2} < \frac{p_2}{p_3} < \ldots .$$

Let now $$a_0 H_{a_0} - a_1 H_{a_1} + a_2 H_{a_2} - \ldots = 0,$$

where $\alpha_0, \alpha_1, \alpha_2, \ldots$ are in descending order of magnitude and differ successively by the same quantity. Then we still have

$$\frac{H_{a_0}}{H_{a_1}} < \frac{H_{a_1}}{H_{a_2}} < \frac{H_{a_2}}{H_{a_3}} < \frac{H_{a_3}}{H_{a_4}} < \ldots,$$

and

$$\frac{a_0}{a_1} < \frac{a_1}{a_2} < \frac{a_2}{a_3} < \ldots .$$

V. Consider

$$\begin{vmatrix} s_a, & s_{a+\alpha}, & s_{a+\beta} \\ s_b, & s_{b+\alpha}, & s_{b+\beta} \\ s_c, & s_{c+\alpha}, & s_{c+\beta}, \ \ldots \end{vmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (\gamma),$$

where s_n denotes the sum of the nth powers of n positive quantities $a, b, c, \ldots$, the number of rows and columns is n, and the rows and columns are so disposed that we have

$$a < b < c < \ldots,$$

and

$$\alpha < \beta < \gamma < \ldots .$$

Now we have

$$s_m - p_1 s_{m-1} + p_2 s_{m-2} - \ldots (-1)^n p_n S_{m-n} = 0.$$

This is the same relation as connects the homogeneous functions of the n quantities. Hence, to get the relation between any $n+1$ of the sums of powers, we have to eliminate in the same manner between the same equations, as we have for the relation between the homogeneous functions of the same degree. Hence the ultimate relation will be the same. Thus we have

$$(s_{a_1}) - (s_{a_2}) + (s_{a_3}) - \ldots = 0,$$

where $$a_1 > a_2 > a_3 > \ldots,$$

and there are $n+1$ terms.

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The immediate consequence is that (γ) is of the same sign as

$$\begin{vmatrix} s_0, & s_1, & s_2, & \cdots \\ s_1, & s_2, & s_3, & \cdots \\ s_2, & s_3, & s_4, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

and this is well known to be positive, which leads to the same rule for (γ) as is given in III.

More generally the same law controls

$$\begin{vmatrix} \sigma_{\alpha+\alpha}, & \sigma_{\alpha+\beta}, & \sigma_{\alpha+\gamma}, & \cdots \\ \sigma_{b+\beta}, & \sigma_{b+\beta}, & \sigma_{b+\gamma}, & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

where $\sigma_r = a_1{}^r x_1 + a_2{}^r x_2 + \ldots + a_n{}^r x_n$.

NOTE ON SCHLAEFLI'S MODULAR EQUATION FOR THE CUBIC TRANSFORMATION.

By Prof. *Cayley*.

THE equation in question, *Crelle*, t. 72 (1870), p. 369, is

$$S^4 + T^4 + 8ST - S^3T^3 = 0,$$

where $S = \dfrac{\sqrt{2}}{\sqrt[4]{(kk')}}, \quad T = \dfrac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}}$;

which must be of course equivalent to the ordinary modular equation

$$u^4 - v^4 + 2uv - 2u^3v^3 = 0,$$

where $u = \sqrt[4]{k}, \quad v = \sqrt[4]{\lambda}$;

the resemblance in form between the two equations, with such different meanings of the S, T in the one case and the u, v in the other, is very noticeable.

Schläfli's equation is

$$\frac{4}{kk'} + \frac{4}{\lambda\lambda'} + \frac{16}{(kk'\lambda\lambda')^{\frac{1}{2}}} - \frac{8}{(kk'\lambda\lambda')^{\frac{1}{4}}} = 0,$$

that is $$kk' + \lambda\lambda' = 2(kk'\lambda\lambda')^{\frac{1}{2}} - 4(kk'\lambda\lambda')^{\frac{3}{4}},$$

or say

$$k^2k'^2 + \lambda^2\lambda'^2 = 4(kk'\lambda\lambda')^{\frac{1}{2}} - 18(kk'\lambda\lambda') + 16(kk'\lambda\lambda')^{\frac{3}{4}}.$$

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To deduce this from the uv-modular equation, we have (*Fund. Nova*, p. 68),

$$(1-u^6)(1-v^6)=(1-u^2v^2)^4,$$

or multiplying each side by u^3v^3, and extracting the fourth root

$$\sqrt{(kk'\lambda\lambda')}=u^3v^3(1-u^2v^2)=x-x^3,$$

if for shortness we write $x=u^2v^2$.

The equation to be proved thus is

$$u^6(1-u^6)+v^6(1-v^6)=4(x-x^3)-18(x-x^2)^3+16(x-x^2)^5,$$

But from the foregoing equation $(1-u^6)(1-v^6)=(1-u^2v^2)^4$, we have

$$u^6+v^6=4u^3v^3-6u^4v^4+4u^6v^6,\ =4x-6x^2+4x^3,$$

and thence $$u^{16}+v^{16}=(4x-6x^2+4x^3)^2-2x^4,$$

and the equation to be proved thus becomes

$$(4x-6x^2+4x^3)-(4x-6x^2+4x^3)^2+2x^4$$
$$=4(x-x^2)-18(x-x^2)^3+16(x-x^2)^5,$$

which is in fact an identity, each side being

$$=4x-22x^2+52x^3-66x^4+48x^5-16x^6.$$

ON THE DIFFERENTIAL EQUATION OF CONFOCAL SPHERO-CONICS.

By *W. Burnside, M.A.*

THE equations of a system of confocal sphero-conics

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=0,$$
$$x^2+y^2+z^2=1,$$
$$(a^2>b^2>c^2),$$

lead, on differentiating and eliminating λ, to the equations

$$\frac{(b^2-c^2)x}{y\,dz-z\,dy}+\frac{(c^2-a^2)y}{z\,dx-x\,dz}+\frac{(a^2-b^2)z}{x\,dy-y\,dx}=0,$$

and $$x\,dx+y\,dy+z\,dz=0.$$

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Introducing now as independent variables the conjugate imaginaries

$$\alpha = \tan \tfrac{1}{2}\theta e^{i\phi}, \quad \beta = \tan \tfrac{1}{2}\theta e^{-i\phi},$$

θ and ϕ being the co-latitude and longitude on the sphere reckoned about the positive end of the z-axis as pole; we have

$$x = \frac{\beta+\alpha}{\beta\alpha+1}, \quad y = i\,\frac{\beta-\alpha}{\beta\alpha+1}, \quad z = \frac{\beta\alpha-1}{\beta\alpha+1};$$

and

$$y\,dz - z\,dy = \frac{i\{(\beta^2-1)\,d\alpha - (\alpha^2-1)\,d\beta\}}{(\beta\alpha+1)^2},$$

$$z\,dx - x\,dz = -\frac{\{(\beta^2+1)\,d\alpha + (\alpha^2+1)\,d\beta\}}{(\beta\alpha+1)^2},$$

$$x\,dy - y\,dx = \frac{2i\{-\beta d\alpha + \alpha d\beta\}}{(\beta\alpha+1)^2}.$$

The second differential equation is necessarily now satisfied identically, and the first becomes, on making the given substitutions

$$d\alpha^2[(a^2-b^2)(\beta^4-1)(\beta\alpha-1) - 2(b^2-c^2)\beta(\beta^2+1)(\beta+\alpha) - 2(c^2-a^2)\beta(\beta^2-1)(\beta-\alpha)]$$
$$= d\beta^2[(a^2-b^2)(\alpha^4-1)(\alpha\beta-1) - 2(b^2-c^2)\alpha(\alpha^2+1)(\alpha+\beta) - 2(c^2-a^2)\alpha(\alpha^2-1)(\alpha-\beta)].$$

The factor $\beta\alpha+1$ is common to the two sides of the equation, and on dividing by it we have

$$d\alpha^2[(a^2-c^2)(\beta^2-1)^2 - (b^2-c^2)(\beta^2-1)^2] = d\beta^2[(a^2-c^2)(\alpha^2-1)^2 - (b^2-c^2)(\alpha^2+1)^2].$$

If now

$$\frac{\sqrt{(a^2-c^2)} - \sqrt{(b^2-c^2)}}{\sqrt{(a^2-c^2)} + \sqrt{(b^2-c^2)}} = k,$$

so that k is a positive proper fraction, and if

$$\alpha = \alpha'\sqrt{k}, \quad \beta = \beta'\sqrt{k},$$

the equation becomes

$$\frac{d\alpha'^2}{(1-\alpha'^2)(1-k^2\alpha'^2)} = \frac{d\beta'^2}{(1-\beta'^2)(1-k^2\beta'^2)},$$

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and therefore $$\alpha = \sqrt{k}\,\mathrm{sn}(u+iv),$$
$$\beta = \sqrt{k}\,\mathrm{sn}(u-iv),$$
where $$(u+iv) \pm (u-iv) = \text{constant}.$$

The two systems of confocal sphero-conics are therefore the curves on the sphere for which the real and imaginary parts of the complex function of position w, given by
$$\alpha = \sqrt{k}\,\mathrm{sn}\,w,$$
are respectively constant.

If points on the sphere are stereographically projected on the tangent plane at $x=0$, $y=0$, $z=1$ from the point $x=0$, $y=0$, $z=-1$; ξ, η being the coordinates in the plane of the projection of the point α, β with respect to axes through $x=0$, $y=0$, $z=1$ parallel to the original axes of x and y, then
$$\xi + i\eta = \alpha;$$
and therefore the plane curves $u=\text{const.}$, $v=\text{const.}$, obtained from
$$\xi + i\eta = \sqrt{k}\,\mathrm{sn}(u+iv),$$
are the stereographic projections of the above system of confocal sphero-conics. This affords a ready method of realising the general form of the curves (cf. Holzmüller, *Isogonale Verwandschaften*, §§ 101, et seq., where figures are given).

Since $$1-\xi^2-\eta^2 = 1-k\,\mathrm{sn}(u+iv)\,\mathrm{sn}(u-iv)$$
$$= \frac{(1-k\,\mathrm{sn}^2 u)\,(1+k\,\mathrm{sn}^2 iv)}{1-k^2\,\mathrm{sn}^2 u\,\mathrm{sn}^2 iv},$$
the curve $v=\frac{1}{2}K'$ is a double circle, separating the two ovals of which each of the other v-curves consist, which ovals the above projective property shews to be inverses of each other with respect to this circle.

The substitutions
$$\alpha' = \frac{\alpha - i}{\alpha + i}, \quad \beta' = \frac{\beta + i}{\beta - i},$$
are equivalent to interchanging the axes of y and z; since if θ', ϕ' are co-latitude and longitude for α', β', we have
$$\theta' = 0, \text{ when } \theta = \tfrac{1}{2}\pi, \ \phi = \tfrac{1}{2}\pi,$$
$$\theta' = \pi, \text{ when } \theta = \tfrac{1}{2}\pi, \ \phi = -\tfrac{1}{2}\pi;$$

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and hence the plane curves $u=\text{const.}$, $v=\text{const.}$, obtained from

$$\xi+i\eta=\frac{\sqrt{k}\,\mathrm{sn}\,w-i}{\sqrt{k}\,\mathrm{sn}\,w+i}=\frac{\mathrm{sn}\,w-\mathrm{sn}\,\frac{1}{2}iK'}{\mathrm{sn}\,w+\mathrm{sn}\,\frac{1}{2}iK'},$$

are again the stereographic projections of a system of confocal sphero-conics, the four foci in this case lying in the equator of the sphere as regards the projection. A figure corresponding to this case is also given at the end of the above mentioned work by Holzmüller.

NOTE ON THE NINTH ROOTS OF UNITY.

By Prof. *Cayley*.

LET θ be a prime ninth root of unity, $\theta^6+\theta^3+1=0$, and write

$$a=\theta+\theta^8,$$
$$b=\theta^2+\theta^7,$$
$$c=\theta^4+\theta^5,$$

then $$a+b+c+\theta^6+\theta^3=\frac{1-\theta^9}{1-\theta}-1,\ =-1,$$

that is $$a+b+c=0.$$

Also $$a^2=b+2,\quad bc=b-1,$$
$$b^2=c+2,\quad ca=c-1,$$
$$c^2=a+2,\quad ab=a-1,$$

whence $$ab+ac+bc=-3,$$
$$abc=-1,$$

and a, b, c are thus the roots of the equation $x^3-3x+1=0$.

we have $$a^2b+b^2c+c^2a=a^2+b^2+c^2=6,$$
$$ab^2+bc^2+ca^2=bc+ca+ab=-3,$$

and thence $$-(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)$$
$$=(b-c)(c-a)(a-b)=-6-3,=-9.$$

The equation $x^3-3x+1=0$ is thus such that $a^2b+b^2c+c^2a$, and consequently any rational function whatever of a, b, c, invariable by the cyclical interchange (abc) of the roots, has a rational value.

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ON A PROPERTY OF PLANE ISOTHERMAL CURVES.

By *W. Burnside.*

A FUNCTION u of x and y, which with its differential coefficients $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ is finite, continuous and one-valued at all points of a given plane area and which everywhere within the area satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

is known to be perfectly determinate when its values at the boundaries are given. The given boundary values must be finite and continuous, but are subjected to no other condition. The function u, thus determined, has no maximum or minimum values inside the area; or in other words all its maximum or minimum points are situated on the boundary. The condition that the boundary values of u are finite and continuous involves that on each separate bounding curve the number of maximum points must be equal to the number of minimum points; or the number of maximum and minimum points on each separate bounding curve must be even. If the contour lines, $u =$ constant, are drawn, their forms in the immediate neighbourhood of each maximum or minimum point will be approximately that of small semicircles enclosing the points. No contour line or part of a contour line can consist of an infinitely small or infinitely large closed curve in consequence of the theorem that u has no maximum or minimum values inside the area. It will be found that when the values of u are given numerically or graphically for a sufficient number of points on the boundary to make the number and approximate position of the maximum and minimum points definite, the above conditions enable the general form of the contour lines and in particular the number and approximate position of their double points to be determined by mere drawing. A linear relation exists between the number of bounding curves of the area, the number of maximum and minimum points on the boundary, and the number of double points on the contour lines, which it is the object of this note to establish. The case of an area bounded by a single closed curve is first considered; and for the case when the curve is an internal boundary so that

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Major P. A. MacMahon, "The theory of perfect partitions of numbers and the compositions of multipartite numbers."

Mr. J. Brill, "On the application of the method of reciprocal polars to statical theorems."

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Since $\phi(a_0+a_1x+a_2x^2)$ is obtained from $\phi(a_0+a_1x)$ when a_1+a_2x is written for a_1, we have in like manner

$$\phi(a_0+a_1x+a_2x^2)=(1-xD_1)^{-1}(1-xD_0)^{-1}\phi a_0,$$

and generally

$$\phi(a_0+a_1x+\ldots+a_nx^n)=(1-xD_{n-1})^{-1}(1-xD_{n-2})^{-1}\ldots(1-xD_0)^{-1}\phi a_0.$$

This is sufficient to give the coefficient of x^n in the development of $\phi(a_0+a_1x+a_2x^2+\ldots)$. Now developing the operating factors and collecting the coefficient of x^n, we get the sum of all the terms, such as

$$D^{\alpha}_{n-1}D^{\beta}_{n-2}\ldots D_0^{\mu}\phi a_0,$$

in which $$\alpha+\beta+\ldots+\mu=n.$$

Everyone of these terms will vanish except those in which (1) the subscripts form an unbroken progression $m, m-1, \ldots, 2, 1, 0$, and (2), supposing the D's arranged in order of the subscripts, none of the exponents $\alpha, \beta, \ldots$, are greater than their successors to the right. The first is a particular case of the second condition. The latter is proved by observing that D^{i}_{m-1} introduces the factor $a_m^{\,i}$, and that $D_m^{i+1}a_m^{\,i}=0$, because a part of the operation D_m^{i+1} is $\left(\frac{\partial}{\partial a_m}\right)^{i+1}$. Thus

$$D_m^{i+1}D^{i}_{m-1}\ldots\phi a_0=0.$$

Moreover, α can only be 0 or 1 because a_n cannot enter the coefficient of x^n with an exponent greater than 1.

For the purpose of "deriving" the coefficient of x^{n+1} from that of x^n, we suppose given all combinations, such as

$$D^{\alpha}_{n-1}D^{\beta}_{n-2}\ldots D_0^{\mu}\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and we want all such combinations as

$$D_n^{\alpha'}D_{n-1}^{\beta'}\ldots D_0^{\nu'}\ \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where $$\alpha+\beta+\ldots+\mu=n,$$

$$\alpha'+\beta'+\ldots+\nu'=n+1.$$

Now since α, α' are either 1 or 0, we see that every term such as (2) can be got from some term such as (1), either (a) by prefixing the D with the next higher subscript or (b) by adding 1 to the exponent of some D. It is sufficient to take the last D for this purpose. For suppose the last symbol of a term is $D_m^{\,i}$ and another is $D_p^{\,k}$ $(p<m)$. Then there will be another term in (1) differing only from this in containing $D_m^{\,i-1}$ and $D_p^{\,k+1}$, which gives the same combination by adding

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1 to the exponent of D_m as is given by the former on adding 1 to the exponent of D_p. The operation (b) gives a zero term when the exponents of D_m, D_{m-1} are equal, in which case there the term contains a_{m+1} but not a_m. Hence we have the rule for deriving. "If m be the highest suffix in the term, operate with D_m only unless a_{m-1} be also contained in the term, and then operate with D_{m-1} also."

Donkin's proof seems a more natural outcome of De Morgan's box and counters analogy (viii) than De Morgan's own proof: and it is worthy of remark how Donkin has met the difficulty in using the calculus of operations which was indicated by De Morgan.

(xi). CAYLEY, A. '*On an extension of Arbogast's method of derivations.*' *Phil. Trans.* 1861, pp. 37–43. *Proc. R. S.*, vol. XI., pp. 42, 3, (No. 233, R. S. Cat.)

Arbogast's method is at least as useful for forming combinations of given degree and weight in the letters (a, b, c, d, ...), the weights of the successive letters being 0, 1, 2, 3, &c., as for its original purpose. The object of this memoir is to investigate a rule for forming all combinations of given weight and degree independently of the others. Incidentally Prof. Cayley indicates a proof of Arbogast's method of derivation which is of the same character as that in (ix).

(xii) ROBERTS, S. '*On Arbogast's calculus of derivations.*' *Q. J. Math.*, vol. IV, 1861, pp. 193–204 (No. 11, R. S. Cat.)

(xiii) ROBERTS, S. '*Notes on Arbogast's calculus of derivations.*' *Q. J. Math.*, vol. VII, 1866, pp. 10–19 (No. 14, R. S. Cat.).*

The first paper (xii) begins with the theorem:

If $$\Pi = a_1 \frac{\partial}{\partial a_0} + 2a_2 \frac{\partial}{\partial a_1} + 3a_3 \frac{\partial}{\partial a_2} + \ldots,$$

then $$\phi(a_0 + a_1 x + a_2 x^2 + \ldots) = e^{x\Pi}\, \phi(a_0) \ldots\ldots\ldots (a).$$

* My first knowledge of these papers was obtained from two letters in which Mr. Roberts was good enough to answer an enquiry of mine. Upon these letters the abstract is largely based. They were the more useful because, as appears from a postcript to the 'Notes,' the original MS. of the Notes was lost in the post, and in the Notes as printed some passages require amplification. It is perhaps unnecessary to add that the modifications in no wise touch the principle of the proof and that I alone am responsible for the opinions expressed.

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"For if $\phi(a_0 + a_1x + \ldots)$ be differentiated with respect to any coefficient a_p, and multiplied by $(p+1)\,a_{p+1}$, we obtain

$$\phi'(a_0 + a_1x + \ldots)^p (p+1)\,a_{p+1}x^p,$$

and consequently

$$\Pi\,\phi(a_0 + a_1x + \ldots) = \frac{\partial}{\partial x}\phi_0(a_0 + a_1x + \ldots):$$

and further

$$\Pi^n\,\phi(a_0 + a_1x + \ldots) = \frac{\partial^n}{\partial x^n}\phi_0(a_0 + a_1x + \ldots),$$

or
$$\Pi^n\,\phi a_0 = \left(\frac{\partial}{\partial O}\right)^n \phi(a_0 + a_1O + \ldots),$$

from which the formula (a) follows by Maclaurin's theorem."

Since $\Pi^n\,\phi(a_0)/(n!)$ is the coefficient of x^n in the development of $\phi(a_0 + a_1x + \ldots)$, it is the same thing as $D^n\,\phi(a_0)$, where D is the symbol of Arbogast's derivation. Hence the title of the paper (xii) which deals with Π and allied operators.*

The last and last but one rule is investigated in the 'Notes' with the help of the following theorem, in which

$$\Pi_r = a_1\frac{\partial}{\partial a_0} + 2a_2\frac{\partial}{\partial a_1} + \ldots + ra_r\frac{\partial}{\partial a_{r-1}},$$

and Π_{r+1} has a corresponding meaning.

The theorem is:

$$\frac{\Pi^n_{r+1}}{n!}\phi a_0$$
$$= \left\{\frac{\Pi_r^n}{n!} + \frac{a_{r+1}\dfrac{\partial}{\partial a_r}}{1}\frac{\Pi_r^{n-1}}{n-1!} + \ldots + \frac{\left(a_{r+1}\dfrac{\partial}{\partial a_r}\right)^p}{p!}\frac{\Pi_r^{n-p}}{n-p!} + \ldots\right\}\phi a_0 \ldots (b)\dagger.$$

* In a paper "on the Multiplication of Partial Differential Operators" (*Phil. Mag.*, 4th Series, vol. xxxiii, 1867), Prof. Sylvester calls this operator, Π, an Arbogastiant (p. 49 footnote).

† The import of the theorem is more fully brought out by writing it in the form

$$\{\Pi_r + (r+1)\,\delta\}^n\,\phi a_0 = [\Pi_r + \delta]^n\,\phi a_0,$$

where δ is for the moment written instead of

$$a_{r+1}\frac{\partial}{\partial a_r}.$$

In each term of the development of $[\ldots]^n$ the operators are rearranged so that all the Π_r act before any of the δ. In the development of $\{\ldots\}^n$ the operators retain their proper sequence: and the theorem is that all the terms by which $\{\ldots\}^n$ exceeds $[\ldots]^n$ produce 0 when the operand is ϕa_0.

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The proof consists in showing that each side is the coefficient of x^n in the expansion of $\phi(a_0 + a_1x + \ldots + a_{r+1}x^{r+1})$. On the left this coefficient is given immediately. On the right it is arranged according to the powers of a_{r+1}. For instance, $\Pi_r^{n-p}\phi a_0/(n-p)!$ is the coefficient of x^{n-p} in the expansion of $\phi(a_0 + a_1x + \ldots + a_rx^r)$. Call this ψa_r. We can re-introduce a_{r+1} by changing a_r into $a_r + a_{r+1}x$. Now the coefficient of x^p in $\psi(a_r + a_{r+1}x)$ is

$$\frac{\left(a_{r+1}\frac{\partial}{\partial a_r}\right)^p}{p!}\psi(a_r) = \frac{\left(a_{r+1}\frac{\partial}{\partial a_r}\right)^p}{p!}\frac{\Pi_r^{n-p}}{(n-p)!}\phi a_0,$$

This is therefore part of the coefficient of x^n $(= x^p x^{n-p})$ in the expansion of $\phi(a_0 + a_1x + \ldots + a_{r+1}x^{r+1})$; namely the part which contains a_{r+1}^p.

The series on the right of (b) may be extended to $n+1$ terms. But if $p(r+1)$ is greater than n, the result of the operation on ϕa_0 is 0, for the presence of a_{r+1}^p implies the presence of $x^{p(r+1)}$. Hence the number of effective terms on the right is not greater than $(n+r+1)/n$.

The theorem (b) may be written

$$\left(\frac{\Pi_{r+1}^n}{n!} - \frac{\Pi_r^n}{n!}\right)\phi a_0$$

$$= \left\{\frac{a_{r+1}\frac{\partial}{\partial a_r}}{1}\frac{\Pi_r^{n-1}}{(n-1)!} + \ldots + \frac{\left(a_{r+1}\frac{\partial}{\partial a_r}\right)^p}{p!}\frac{\Pi_r^{n-p}}{n-p!} + \ldots\right\}\phi a_0$$

$$= S_r\frac{\Pi_{r+1}^{n-1}}{(n-1)!}\phi a_0,$$

where S_r indicates that for each value of $p\,(=1, 2, \ldots)$, $\left(a_{r+1}\frac{\partial}{\partial a_r}\right)^{p-1}$ is to be changed into $\frac{1}{p}\left(a_{r+1}\frac{\partial}{\partial a_r}\right)^p$ in the development of $\Pi_{r+1}^{n-1}/(n-1)!$. S_r is in fact the operator $\left(\frac{\partial}{\partial a_{r+1}}\right)^{-1}\frac{\partial}{\partial a_r}$, that is to say, derivation with respect to a_r.

Now the left side is that part of the coefficient of x^n in the expansion of $\phi(a_0 + a_1x + \ldots)$ which actually contains a_{r+1}, but no a with suffix greater than $r+1$. And the theorem expresses that this part is obtained by derivation with respect to a_r from $\Pi_{r+1}^{n-1}\phi a_0/(n-1)!$. But $\Pi_{r+1}^{n-1}\phi a_0/(n-1)!$ is the coefficient of x^{n-1} in the expansion of $\phi(a_0 + a_1x + \ldots a_{r+1}x^{r+1})$ and contains no a beyond a_{r+1}, so that the derivation is always with respect to the last or last but one letter.

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This proof differs from its predecessors not only in considering derivation as the divided Π, but, more fundamentally, in the analysis of the coefficient of x^n. If we call the greatest suffix in a term, the *extent* of the term, Mr. Roberts' theorem is that all terms of weight n and extent $r+1$ can be found by derivation with respect to a_r from the terms of weight $n-1$ and of extent not exceeding $r+1$, or, what is equivalent, of extents $r+1$, r.

It is of some interest to prove the theorem in Donkin's notation (x). The part of the expansion of $\phi(a_0+a_1x+\ldots)$, the extents of which do not exceed $r+1$, is

$$(1-xD_r)^{-1}(1-xD_{r-1})^{-1}\ldots(1-xD_0)^{-1}\phi a_0.$$

Hence the part which is of extent $r+1$ is

$$\{(1-xD_r)^{-1}-1\}(1-xD_{r-1})^{-1}\ldots(1-xD_0)^{-1}\phi a_0$$
$$=xD_r(1-xD_r)^{-1}(1-xD_{r-1})^{-1}\ldots(1-xD_0)^{-1}\phi a_0,$$

and, remembering that the exponent of x in any term is the weight of the coefficient, this gives Mr. Roberts' theorem immediately.

(xiv) De Morgan, A. '*On a problem in Annuities and on Arbogast's method of Development.*' *Journal of Institute of Actuaries*, vol. xii, pp. 206–212, 1865.

An amplification of (ix) with examples. The proof is extended to the general case $\phi(t+ax+bx^2+\ldots)$.

(xv) Thomson, J. J. '*An Extension of Arbogast's method of derivation.*' *Messenger*, vol. vii, pp. 142, 143. (1878).

(xvi) Cayley, A. '*Note on Arbogast's Method of Derivations.*' *Messenger*, vol. vii. p. 158. (1878).

(xvii) David, G. '*Applications de la dérivation d'Arbogast à la solution de la partition des nombres et à d'autres problèmes.*' *Résal Journal*, *Sér.* 3, vol. viii, pp. 61–72. (1882).

M. David devotes the first paragraph of his memoir to a succinct account of Arbogast's method, sufficiently full to enable the reader to understand the applications which follow.

(xviii) David, G. '*Applications de la dérivation d'Arbogast. Formule générale pour le changement de la variable indépendente.*' *Jordan, Journal*, 4. *Sér.* vol. iii. 1887, pp. 53–62.

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NOTE ON CONGRUENCIES OF LINES.

By *J. P. Johnston, M.A.*

THE following proof of the proposition, that every congruency of lines may, in general, be regarded as the system of bitangents of a certain surface, is perhaps worth recording.

Let the equations of the congruency be

$$\left.\begin{aligned} x &= ly + mz \\ w &= l'y + m'z \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$\left.\begin{aligned} \phi\,(l, m, l', m') &= 0 \\ \psi\,(l, m, l', m') &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Then if x, y, z, w be the coordinates of a point on the line (1), where it is intersected by a consecutive line $(l + \delta l,\ m + \delta m,\ l' + \delta l',\ m' + \delta m')$ of the congruency, we have

$$y\delta l + z\delta m = 0,$$
$$y\delta l' + z\delta m' = 0,$$

we also have

$$\frac{\partial \phi}{\partial l}\delta l + \frac{\partial \phi}{\partial m}\delta m + \frac{\partial \phi}{\partial l'}\delta l' + \frac{\partial \phi}{\partial m'}\delta m' = 0,$$
$$\frac{\partial \psi}{\partial l}\delta l + \frac{\partial \psi}{\partial m}\delta m + \frac{\partial \psi}{\partial l'}\delta l' + \frac{\partial \psi}{\partial m'}\delta m' = 0.$$

Eliminating δl, δm, $\delta l'$, $\delta m'$, from these four equations we get

$$\begin{vmatrix} z\dfrac{\partial \phi}{\partial l} - y\dfrac{\partial \phi}{\partial m}, & z\dfrac{\partial \phi}{\partial l'} - y\dfrac{\partial \phi}{\partial m'} \\ z\dfrac{\partial \psi}{\partial l} - y\dfrac{\partial \psi}{\partial m}, & z\dfrac{\partial \psi}{\partial l'} - y\dfrac{\partial \psi}{\partial m'} \end{vmatrix} = 0,$$

or

$$y^2\frac{\partial(\phi, \psi)}{\partial(m, m')} - yz\left(\frac{\partial(\phi, \psi)}{\partial(m, l')} + \frac{\partial(\phi, \psi)}{\partial(l, m')}\right) + z^2\frac{\partial(\phi, \psi)}{\partial(l, l')} = 0 \ldots (3).$$

Consequently every line of a congruency is intersected in two points by a pair of consecutive lines, from which it follows that the lines of a congruency are bitangents to a certain surface. The equation of this surface is obtained by eliminating l, m, l', m', from the equations (1), (2), and (3).

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It is geometrically evident that the locus of two points of intersection which coincide is a nodal line on the surface. The condition for the coincidence of the two points is

$$4\frac{\partial(\phi,\psi)}{\partial(l,l')}\frac{\partial(\phi,\psi)}{\partial(m,m')}=\left(\frac{\partial(\phi,\psi)}{\partial(m,l')}+\frac{\partial(\phi,\psi)}{\partial(l,m')}\right)^2 \quad \ldots(4).$$

Therefore the resultant of the equations (1), (2), and (4) with respect to l, m, l', m', equated to zero, intersects the bitangential surface in a nodal line.

THE THEORY OF PERFECT PARTITIONS OF NUMBERS AND THE COMPOSITIONS OF MULTIPARTITE NUMBERS.

By *Major P. A. MacMahon, R.A., F.R.S.*

1. In the *Quarterly Journal of Mathematics* for 1886 I brought to notice certain partitions of numbers which possess a peculiar and interesting property with respect to numbers less than the number partitioned. I called these at the time 'perfect partitions' and the nomenclature seems to be sufficiently appropriate. The definition is as follows:—

'A perfect partition of a number is one which contains one and only one partition of every lesser number.'

The word number is for the present taken to mean 'positive and integral number.' In some cases hereafter the restriction may be removed so as to bring under view all positive, zero, and negative integers.

Other partitions possess a very similar property. These may be termed provisionally subperfect, and may be defined as follows:—

'A subperfect partition of a number is one which contains one and only one partition of every lesser number if it be permissible to regard the several parts as affected with *either* the positive or negative sign.

Having recently discovered the law by which these partitions may in all cases be enumerated I here give an account of the investigation.

Every assemblage of the parts of a partition, or say every collection of the parts involving one or more parts, is a partition of some number. If the partition be 'perfect' no two of these collections can be partitions of the same number,

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and, moreover, the number of such collections must be equal to the partitioned number. Consider one of the perfect partitions of 7, viz.

$$(4111);$$

that it is perfect is at once obvious.

2. Before proceeding to the general case of enumeration it is convenient to first solve the problem for certain special numbers included in the formula

$$a^\alpha - 1,$$

where a is a prime number; these numbers when increased by unity are powers of primes. It will be shewn that this problem is identical with that of the enumeration of the compositions of numbers. Partitions in which account is also taken of the order of occurrence of the parts are termed by Glaisher 'compositions.'

Not only these numbers but all numbers have clearly one perfect partition composed wholly of unit parts. Represent this solution by the identity

$$\frac{1-x^{a^\alpha}}{1-x} = 1 + x + x^2 + \ldots + x^{a^\alpha - 1}.$$

If the right-hand side be factorized into expressions of the form

$$1 + x^q + x^{2q} + \ldots + x^{pq}$$

the arithmetical interpretation of the factorization proves the existence of a perfect partition of the number $a^\alpha - 1$ in correspondence therewith.

This factorization is most conveniently studied by means of the left-hand side

$$\frac{1-x^{a^\alpha}}{1-x},$$

for any exhibition of this expression in the form

$$\frac{1-x^{a^\alpha}}{1-x^p} \cdot \frac{1-x^p}{1-x^q} \cdot \frac{1-x^q}{1-x^r} \cdots \frac{1-x^u}{1-x},$$

wherein $p, q, r, \ldots, 1$ are factors of

$$a^\alpha, p, q, \ldots u \text{ respectively,}$$

yields a factorization of the form required. Hence we are led to the consideration of the factors of the number a^α. The factorization of the number a^α is the same problem as the partitioning of the number α. In the first place take a

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particular value of a, viz. 4. We have for one case of factorization

$$\frac{1-x^{a^4}}{1-x^a}\cdot\frac{1-x^a}{1-x}.$$

Let us agree for brevity to denote such a fraction as $\frac{1-x^{a^p}}{1-x^{a^q}}$ by $p-q$, which, observe, is the difference of the power indices of a in the numerator and denominator.

Then we may write

$$\frac{1-x^{a^4}}{1-x^a}\cdot\frac{1-x^a}{1-x}=3,\ 1,$$

3, 1 being of necessity a partition of 4.

There is also the factorization represented by

$$\frac{1-x^{a^4}}{1-x^{a^3}}\cdot\frac{1-x^{a^3}}{1-x}=1,\ 3,$$

where the same numbers 1, 3 occur, but in a different order. Whatever case of factorization be exhibited we must always obtain on the right a succession of numbers constituting a partition of 4. Since, moreover, the same numbers may occur in various orders it is more appropriate to speak of 'compositions' in lieu of partitions. The whole of the factorizations are

$$\frac{1-x^{a^4}}{1-x}=4,$$

$$\frac{1-x^{a^4}}{1-x^a}\cdot\frac{1-x^a}{1-x}=3,\ 1,$$

$$\frac{1-x^{a^4}}{1-x^{a^3}}\cdot\frac{1-x^{a^3}}{1-x}=1,\ 3,$$

$$\frac{1-x^{a^4}}{1-x^{a^2}}\cdot\frac{1-x^{a^2}}{1-x}=2,\ 2,$$

$$\frac{1-x^{a^4}}{1-x^{a^2}}\cdot\frac{1-x^{a^2}}{1-x^a}\cdot\frac{1-x^a}{1-x}=2,\ 1,\ 1,$$

$$\frac{1-x^{a^4}}{1-x^{a^3}}\cdot\frac{1-x^{a^3}}{1-x^a}\cdot\frac{1-x^a}{1-x}=1,\ 2,\ 1,$$

$$\frac{1-x^{a^4}}{1-x^{a^3}}\cdot\frac{1-x^{a^3}}{1-x^{a^2}}\cdot\frac{1-x^{a^2}}{1-x}=1,\ 1,\ 2,$$

$$\frac{1-x^{a^4}}{1-x^{a^3}}\cdot\frac{1-x^{a^3}}{1-x^{a^2}}\cdot\frac{1-x^{a^2}}{1-x^a}\cdot\frac{1-x^a}{1-x}=1,\ 1,\ 1,\ 1.$$

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We have an instance of factorization corresponding to every composition of the number 4.

Every case of factorization is of necessity represented by a different composition of 4, and every composition of 4 is necessarily in correspondence with a separate factorization. Hence the number of factorizations of the nature considered must be equal to the number of compositions of 4. This means that the number of perfect partitions of the number $a^4 - 1$ is equal to the number of compositions of the number 4.

Returning to the more general case of the factorization of

$$\frac{1 - x^{a^2}}{1 - x}$$

it is clear that a factorization

$$\frac{1 - x^{a^\alpha}}{1 - x^{a^{\alpha_1}}} \cdot \frac{1 - x^{a^{\alpha_1}}}{1 - x^{a^{\alpha_2}}} \cdot \frac{1 - x^{a^{\alpha_2}}}{1 - x^{a^{\alpha_3}}} \cdots$$

will yield the composition $\alpha - \alpha_1$, $\alpha_1 - \alpha_2$, $\alpha_2 - \alpha_3$, ... of the number α. Moreover, any composition A, B, C, ... of the number α can be put into the form

$$\alpha - (\alpha - A), \quad \alpha - A - (\alpha - A - B),$$

$$\alpha - A - B - (\alpha - A - B - C), \ldots,$$

and will thus correspond to a factorization

$$\frac{1 - x^{a^\alpha}}{1 - x^{a^{\alpha - A}}} \cdot \frac{1 - x^{a^{\alpha - A}}}{1 - x^{a^{\alpha - A - B}}} \cdot \frac{1 - x^{a^{\alpha - A - B}}}{1 - x^{a^{\alpha - A - B - C}}} \cdots,$$

Hence there is a one-to-one correspondence between the factorizations of $\dfrac{1 - x^{a^\alpha}}{1 - x}$ and the compositions of the number α, *i.e.* between the perfect partitions of the number $a^\alpha - 1$ and the compositions of the number α.

Hence this special problem is solved in the form:

A number of the form $a^\alpha - 1$, where a is a prime, possesses the same number of perfect partitions as the number α possesses compositions.

The number of compositions of α is, as is well known, equal to $2^{\alpha - 1}$.

3. For the comprehension of the general case it is necessary to consider the partitions and compositions of multipartite numbers.

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An ordinary or unipartite number may be regarded as representing a succession of things, experiences, intervals or operations of any kinds, but, in any one number, of the same kind. It is an entity, and when partitioned its parts are entities of the same kind. The succession, being of similar things, is appropriately designated by a single number. Should however the succession be of things not all similar, the succession demands a succession of numbers for its adequate expression. Such succession of numbers may be viewed as a single entity, and provisionally be termed a multipartite number. When such a succession of things is distributed or (the same thing) when such a multipartite number is partitioned, the things in any portion of the distribution or any part of the partition may be regarded as an entity of the same character as the original entity. The parts into which the multipartite number is partitioned should be regarded as being themselves multipartite numbers. An entity such as 3 hours 43 minutes 37 seconds would be represented by a tripartite number 3,43,37, and would be partitioned into parts each tripartite. If one part were 1 hour, which might apparently be denoted by a unipartite number 1, still the part would be regarded as tripartite and would be represented by 1, 0, 0. It is thus clear that certain entities may be adequately represented by multipartite numbers, and that such numbers may be partitioned into multipartite parts (to the same order of multiplicity) which also are representative of entities of the same nature. It is in some respects convenient to denote a multipartite number involving a succession of unipartite numbers α, β, γ, ..., in the notation*

$$(\alpha\beta\gamma\ldots).$$

Observe that regard must be paid to the order of the succession of these unipartite numbers which go to make up the multipartite. The numbers $(\alpha\beta\gamma\ldots)$, $(\beta\alpha\gamma\ldots)$ must be viewed as distinct multipartites. A multipartite number may contain some zeros amongst the succession of numbers of which it is formed, but in the present theory must contain at least one non-zero number. In fact a zero part or number implies a part or number composed altogether of zeros, and as such is here excluded. Thus a part in a partition may contain any number of zeros less than the full number possible. In forming a partition of the number $(\alpha\beta\gamma\ldots)$ any part of α may be taken with any parts of β, γ, &c., ..., zeros not excluded but subject to the condition given above, to form a multipartite part.

* See *Phil. Trans. of the Royal Society*, Vol. 181 (1890), A, pp. 481—536.

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Ex. gr. The partitions of the multipartite number $(\overline{111})$ are $(\overline{111})$, $(\overline{110}, \overline{001})$, $(\overline{101}, \overline{010})$, $(\overline{011}, \overline{100})$, $(\overline{100}, \overline{010}, \overline{001})$, they are five in number, and these, by attending to the order of the parts, give rise to thirteen compositions.

Leaving this short digression I return to the main subject.

4. The number of perfect partitions of the number $a^\alpha b^\beta c^\gamma \ldots - 1$ (where $a, b, c, \ldots$ are primes) depends upon the factorization of the expression

$$\frac{1 - x^{a^\alpha b^\beta c^\gamma \ldots}}{1 - x}$$

into factors of the form $\dfrac{1 - x^q}{1 - x^p}$, where p is a divisor of q.

Suppose such a factorization to be

$$\frac{1 - x^{a^\alpha b^\beta c^\gamma \ldots}}{1 - x^{a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots}} \cdot \frac{1 - x^{a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots}}{1 - x^{a^{\alpha_2} b^{\beta_2} c^{\gamma_2} \ldots}} \cdots \frac{1 - x^{a^{\alpha_s} b^{\beta_s} c^{\gamma_s} \ldots}}{1 - x}.$$

This, in analogy with what has gone before, may be denoted by the symbolism

$$\{(\alpha - \alpha_1, \beta - \beta_1, \gamma - \gamma_1, \ldots), (\alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \ldots), \ldots (\alpha_s \beta_s \gamma_s \ldots)\},$$

which may be viewed as a composition of the multipartite number

$$\{(\alpha\beta\gamma \ldots)\}.$$

Every factorization of the required form is necessarily symbolized by one composition of this multipartite number, and, further, every composition must be the symbol of some one factorization. Hence there is a one-to-one correspondence between the required factorizations of

$$\frac{1 - x^{a^\alpha b^\beta c^\gamma \ldots}}{1 - x}$$

and the composition of the multipartite number $\{(\alpha\beta\gamma \ldots)\}$. Hence their number must be equal, and we have the theorem: "The number $a^\alpha b^\beta c^\gamma \ldots - 1$ possesses as many perfect partitions as the multipartite number $\{(\alpha\beta\gamma \ldots)\}$ possesses compositions."

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5. It is interesting to identify the various perfect partitions with the particular compositions to which they correspond. The perfect partition of the number $a^\alpha - 1$ which arises from the factorization

$$\frac{1-x^{a^\alpha}}{1-x^{a^{\alpha_1}}} \cdot \frac{1-x^{a^{\alpha_1}}}{1-x^{a^{\alpha_2}}} \cdots \frac{1-x^{a^{\alpha_{p-1}}}}{1-x}$$

consists of

$a^{\alpha-\alpha_1} - 1$ parts each equal to a^{α_1},
$a^{\alpha_1-\alpha_2} - 1$,, ,, a^{α_2},
.....
$a^{\alpha_{p-1}} - 1$,, ,, 1,

while the corresponding composition of the number α is as above shewn

$$(\alpha - \alpha_1,\ \alpha_1 - \alpha_2,\ \ldots,\ \alpha_{p-1}).$$

We can therefore derive at once the composition from the partition or the partition from the composition.

Understanding by *frequencies* the numbers

$$a^{\alpha-\alpha_1} - 1,\ a^{\alpha_1-\alpha_2} - 1,\ \ldots$$

appertaining to the various different parts of the perfect partition, we see that the frequencies are, in some order, the same for all compositions derived by permutation from the same partition of the number α.

The number of perfect partitions which possess in some order the same frequencies is the same as the number of such compositions. All perfect partitions which possess, in some order, the same frequencies form a group in correspondence with some partition of the number α. Perfect partitions having the same frequencies have also the same number of parts. It is convenient to consider the perfect partitions as divided into groups. As an illustration I give now the classification of the perfect partitions of the number 15. Since 15 is $2^4 - 1$ they will be associated with the compositions of the number 4.

Composition.	Frequency.	Perfect Partition.	Group.
(4)	15,	(1^{15})	(4),
(31)	7, 1,	$(2^7 1)$	(31),
(13)	1, 7,	$(8 1^7)$	
(22)	3, 3,	$(4^3 1^3)$	(22),
(211)	3, 1, 1,	$(4^3 21)$	(211).
(121)	1, 3, 1,	$(8 2^3 1)$	
(112)	1, 1, 3,	$(8 4 1^3)$	
(1111)	1, 1, 1, 1,	(8421)	(1111).

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A perfect partition of the number $a^a - 1$ has in general

$$a^{a-a_1} + a^{a_1-a_2} + \ldots + a^{a_{p-1}} - p \text{ parts},$$

and the associated composition p parts.

6. The subject of the minimum value of this number is now presented for enquiry.

The expression

$$a^{a-a_1} + a^{a_1-a_2} + \ldots + a^{a_{p-1}},$$

is a sum of terms of which the continued product, being a^a, is constant. Suppose in the first place that the number p is constant so that the composition has a fixed number p of parts. The number of parts in all perfect partitions of the same group is as we have seen constant. How then does this number vary as we change the group while keeping the number of parts in the composition constant. The expression

$$a^{a-a_1} + a^{a_1-a_2} + \ldots + a^{a_{p-1}},$$

where p is constant, has a certain minimum value for a certain partition of the number which defines the group to which the perfect partition belongs. Within this group the number denoted by the expression has a constant value.

Suppose any two unequal terms of the expression

$$a^{a-a_1},\ a^{a_1-a_2},$$

to be replaced by the two terms

$$(a^{a-a_1} . a^{a_1-a_2})^{\frac{1}{2}},\ (a^{a-a_1} . a^{a_1-a_2})^{\frac{1}{2}},$$

viz., by $a^{\frac{1}{2}(a-a_2)},\ a^{\frac{1}{2}(a-a_2)},$

then the sum of the terms is diminished while their *product* remains constant.

Hence, so long as the sum contains two unequal terms the sum may be diminished without altering the constant product. Hence, the sum will be least when all the terms are equal. If then the number a possesses a partition containing p equal parts, this partition will denote the group which yields perfect partitions having the fewest parts for the given value of p.

The fewest number of parts is in this case seen to be

$$p a^{\frac{a}{p}} - p,$$

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and the perfect partition will consist of

$$a^{\frac{\alpha}{p}} - 1 \text{ parts each equal to } a^{\alpha - \frac{\alpha}{p}},$$

$$a^{\frac{\alpha}{p}} - 1 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad a^{\alpha - \frac{2\alpha}{p}},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a^{\frac{\alpha}{p}} - 1 \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad 1,$$

and the associated composition of α is $\frac{\alpha}{p}$ repeated p times.

A number however does not in general possess a partition of this nature, so that a further examination is necessary.

Since we cannot always partition a number α into p equal parts where p is a fixed number, we must so partition the number so that the parts shall be as nearly equal as possible. It is always possible to partition a number α into p parts in such wise that only two different parts occur. We can further take the two parts to be respectively equal to the integers next above and next below the fraction $\alpha \div p$. It is postulated here that α is not of the form 0 mod. p. The partition is thus

$$\frac{\alpha + \xi}{p} \text{ repeated } \eta \text{ times and } \frac{\alpha - \eta}{p} \text{ repeated } \xi \text{ times,}$$

where $\frac{\alpha + \xi}{p}$ and $\frac{\alpha - \eta}{p}$ are the integers next above and next below $\alpha \div p$, and of necessity $\xi + \eta = p$.

This is the partition of α for a fixed value of p which yields perfect partitions with the fewest number of parts.

This fewest number is thus

$$\eta a^{\frac{\alpha - \xi}{p}} + \xi a^{\frac{\alpha - \eta}{p}} - p,$$

and all perfect partitions associated with compositions of the partition

$$\frac{\alpha + \xi}{p} \text{ repeated } \eta \text{ times and } \frac{\alpha - \eta}{p} \text{ repeated } \xi \text{ times}$$

possess this number of parts, which number is the least possible for a given value of p.

This group of perfect partitions is composed of $\frac{(\xi + \eta)!}{\xi!\,\eta!}$ members, since the partition of the group possesses this

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number of compositions. The perfect partitions under discussion may be all written down. The first is

$$\begin{aligned}
&a^{\frac{\alpha+\xi}{p}}-1 \text{ parts each equal to } a^{\alpha-\frac{\alpha+\xi}{p}},\\
&a^{\frac{\alpha+\xi}{p}}-1 \quad ,, \quad ,, \quad ,, \quad a^{\alpha-2\frac{\alpha+\xi}{p}},\\
&\cdots\cdots\cdots

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Next when α is not $\equiv 0$, mod. p. Consider the expression arrived at, viz:

$$\eta a^{\frac{\alpha+\xi}{p}} + \xi a^{\frac{\alpha+\eta}{p}} - p.$$

Where $\alpha \equiv -\xi$ mod. p. $\equiv \eta$, mod. p,

and $\xi + \eta = p$.

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Put $\alpha + \xi = mp$, $\alpha - \eta = (m-1)p$,

and $$u_p = \eta a^{\frac{\alpha+\xi}{p}} + \xi a^{\frac{\alpha-\eta}{p}} - p,$$

$$= (\alpha - mp + p)\, a^m + (mp - \alpha)\, a^{m-1} - p.$$

Then

$$u_p - u_{p+1} = \{(m-1)\alpha - m\}\, a^{m-1} + 1.$$

Since the least value of m is 2 and a is at least 2, the expression on the right is always a positive integer.

Hence u_{p+1} is always less than u_p, and the greater the number of parts in a composition the less the number of parts in the associated perfect partition.

7. Turning to the general case, the perfect partition of the number $a^\alpha b^\beta c^\gamma \ldots - 1$ which corresponds to the factorization

$$\frac{1 - x^{a^\alpha b^\beta c^\gamma \ldots}}{1 - x^{a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots}} \cdot \frac{1 - x^{a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots}}{1 - x^{a^{\alpha_2} b^{\beta_2} c^{\gamma_2} \ldots}} \cdots \frac{1 - x^{a^{\alpha_{p-1}} b^{\beta_{p-1}} c^{\gamma_{p-1}} \ldots}}{1 - x},$$

is

$a^{\alpha - \alpha_1} b^{\beta - \beta_1} c^{\gamma - \gamma_1} \ldots - 1$ parts each equal to $a^{\alpha_1} b^{\beta_1} c^{\gamma_1} \ldots$,

$a^{\alpha_1 - \alpha_2} b^{\beta_1 - \beta_2} c^{\gamma_1 - \gamma_2} \ldots - 1$ „ „ „ $a^{\alpha_2} b^{\beta_2} c^{\gamma_2} \ldots$,

.....

$a^{\alpha_{p-1}} b^{\beta_{p-1}} c^{\gamma_{p-1}} - 1$ „ „ „ 1,

and the associated composition of the multipartite number $\{(\alpha\beta\gamma)\}$ ... is

$$\{(\alpha - \alpha_1, \beta - \beta_1, \gamma - \gamma_1, \ldots \alpha_1 - \alpha_2, \beta_1 - \beta_2, \gamma_1 - \gamma_2, \ldots\ldots$$
$$\alpha_{p-1}, \beta_{p-1}, \gamma_{p-1}, \ldots)\}.$$

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There is a one-to-one correspondence between the perfect partitions and the compositions, and the former are derivable from the latter and *vice versâ*.

For compositions of the same partition the *frequencies* of the perfect partitions are, neglecting order, unaltered. We may generally therefore divide the perfect partitions with groups having reference to the partitions of the ruling multipartite number.

To take a concrete example, write down all the perfect partitions of 244, which is $7^3.5-1$, as follows:—

Multipartite Number.

$(\overline{21})$.

Partition	Composition	Perfect Partition.
$(\overline{21})$	$(\overline{21})$	(1^{244})
$(\overline{20}\ \overline{01})$	$(\overline{20}\ \overline{01})$	$(5^{48}.1^4)$
	$(\overline{01}\ \overline{20})$	$(49^4.1^{48})$
$(\overline{11}\ \overline{10})$	$(\overline{11}\ \overline{10})$	$(7^{34}.1^6)$
	$(\overline{10}\ \overline{11})$	$(35^6.1^{34})$
$(\overline{10}\ \overline{10}\ \overline{01})$	$(\overline{10}\ \overline{10}\ \overline{01})$	$(35^6.5^6.1^4)$
	$(\overline{10}\ \overline{01}\ \overline{10})$	$(35^6.7^4.1^6)$
	$(\overline{01}\ \overline{10}\ \overline{10})$	$(49^4.7^6.1^6)$

Observe that corresponding to a part (xy) in a composition, the perfect partition involves a repetitional number $7^x.5^y-1$.

8. The general expression for the number of parts in a perfect partition is

$$a^{\alpha-\alpha_1}b^{\beta-\beta_1}c^{\gamma-\gamma_1}\ldots + a^{\alpha_1-\alpha_2}b^{\beta_1-\beta_2}c^{\gamma_1-\gamma_2}\ldots +\ldots$$
$$+ a^{\alpha_{p-1}}b^{\beta_{p-1}}c^{\gamma_{p-1}}\ldots - p.$$

Consider a component portion of this, viz: the expression

$$a^{\alpha-\alpha_1}b^{\beta-\beta_1}c^{\gamma-\gamma_1}\ldots - 1,$$

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which may be put into the form

$$(1+a-1)^{\alpha-\alpha_1}(1+b-1)^{\beta-\beta_1}(1+c-1)^{\gamma-\gamma_1}\ldots-1,$$

and is thus seen to be

$$>(\alpha-\alpha_1)\,(a-1)+(\beta-\beta_1)\,(b-1)+(\gamma-\gamma_1)\,(c-1)+\ldots$$

(for a, b, c, ... are necessarily each >1), unless all but one of the numbers $\alpha-\alpha_1$, $\beta-\beta_1$, $\gamma-\gamma_1$, ... are equal to zero and the remaining number equal to unity, when the relation is one of equality.

Hence, from a summation it is clear that in general the expression

$$a^{\alpha-\alpha_1}b^{\beta-\beta_1}c^{\gamma-\gamma_1}\ldots+a^{\alpha_1-\alpha_2}b^{\beta_1-\beta_2}c^{\gamma_1-\gamma_2}\ldots+\ldots+a^{\alpha_{p-1}}b^{\beta_{p-1}}c^{\gamma_{p-1}}\ldots-p,$$

$$>\alpha\,(a-1)+\beta\,(b-1)+\gamma\,(c-1)+\ldots.$$

When however in *each* product

$$a^{\alpha_s-\alpha_{s+1}}b^{\beta_s-\beta_{s+1}}c^{\gamma_s-\gamma_{s+1}}\ldots,$$

all but one of the numbers $\alpha_s-\alpha_{s+1}$, $\beta_s-\beta_{s+1}$, $\gamma_s-\gamma_{s+1}$, ... are equal to zero, and the remaining number equal to unity, then the relation becomes one of equality.

Hence the least value of the expression for the number of parts in a perfect partition is

$$\alpha\,(a-1)+\beta\,(b-1)+\gamma\,(c-1)+\ldots.$$

Then also $p=\alpha+\beta+\gamma+\ldots$, and the composition is

$$\left(\frac{\alpha}{1000\ldots}\,\frac{\beta}{0100\ldots}\,\frac{\gamma}{0010\ldots}\right),$$

or else some other composition obtained by permutation of its parts.

There thus exists a group of perfect partitions the members of which possess the same number of parts, which number is less than the number of parts in any perfect partition not comprised in the group.

The number of partitions in the group is obviously

$$\frac{(\alpha+\beta+\gamma+\ldots)\,!}{\alpha\,!\,\beta\,!\,\gamma\,!\ldots},$$

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and one such perfect partition will consist of

$a-1$ parts each equal to $a^{\alpha-1}b^{\beta}c^{\gamma}\ldots$,

$a-1$ „ „ „ $a^{\alpha-2}b^{\beta}c^{\gamma}\ldots$,

.....

$a-1$ „ „ „ $a^{\alpha-\alpha}b^{\beta}c^{\gamma}\ldots$.

$b-1$ „ „ „ $a^{\alpha}b^{\beta-1}c^{\gamma}\ldots$,

.....

$b-1$ „ „ „ $a^{\alpha}b^{\beta-2}c^{\gamma}\ldots$,

.....

$b-1$ „ „ „ $a^{\alpha}b^{\beta-\beta}c^{\gamma}\ldots$,

$c-1$ „ „ „ $a^{\alpha}b^{\beta}c^{\gamma-1}\ldots$,

$c-1$ „ „ „ $a^{\alpha}b^{\beta}c^{\gamma-\gamma}\ldots$,

.....

$c-1$ „ „ „ $a^{\alpha}b^{\beta}c^{\gamma-\gamma}\ldots$.

&c. &c.

The remaining perfect partitions of the group being obtained by taking the frequencies in the left-hand column in the other orders. The determination of the least number of parts in a perfect partition which corresponds to a composition with a given number p of parts is a more difficult matter into which I do not at present enter.

9. The generating function for the enumeration of the compositions of the multipartite number $(\alpha\beta\gamma\ldots)$ is easily obtained.

If X_r be the sum of the homogeneous products of order r of the quantities $x_1, x_2, x_3, \ldots$ the number in question is the coefficient of $x_1^{\alpha}x_2^{\beta}x_3^{\gamma}\ldots$ in the ascending expansion of the algebraic fraction

$$\frac{X_1+X_2+X_3+\ldots}{1-X_1-X_2-X_3-\ldots},$$

which is

$$\frac{(1-x_1.1-x_2.1-x_3\ldots)^{-1}-1}{1-\{(1-x_1.1-x_2.1-x_3\ldots)^{-1}-1\}},$$

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or $$\frac{\Sigma x_1 - \Sigma x_1 x_2 + \Sigma x_1 x_2 x_3 - \ldots}{(1-x_1 . 1-x_2 . 1-x_3 \ldots) - \Sigma x_1 + \Sigma x_1 x_2 - \Sigma x_1 x_2 x_3 + \ldots},$$

or $$\frac{\Sigma x_1 - \Sigma x_1 x_2 + \Sigma x_1 x_2 x_3 - \ldots}{1 - 2(\Sigma x_1 - \Sigma x_1 x_2 + \Sigma x_1 x_2 x_3 - \ldots)}.$$

As a particular case the compositions of the number α are given by the fraction

$$\frac{x_1}{1-2x_1},$$

which is a well-known result, shewing that the number of compositions is $2^{\alpha-1}$.

For the bipartite numbers $(\alpha\beta)$ the generating function is

$$\frac{x_1 + x_2 - x_1 x_2}{1 - 2(x_1 + x_2 - x_1 x_2)}.$$

The coefficient of $x_1^{\alpha} x_2^{\beta}$ in the expansion is

$$2^{\alpha+\beta-1} \frac{(\alpha+\beta)!}{\alpha!\beta!} - 2^{\alpha+\beta-2} \frac{(\alpha+\beta-1)!}{(\alpha-1)!(\beta-1)!} + 2^{\alpha+\beta-3} \frac{(\alpha+\beta-2)!}{(\alpha-2)!(\beta-2)!} - \ldots,$$

which therefore represents alike the number of compositions of $(\alpha\beta)$ and the number of perfect partitions of the number $a^{\alpha}b^{\beta} - 1$.

10. It may be added that any perfect partition of any number gives rise directly to an unlimited number of perfect partitions of higher numbers.

For this purpose the repetitional index of the leading or highest part may be increased to any extent, or a part equal to the sum of *all* the parts with unity added may be prefixed. Regarded from this point of view, let A_p denote the number of perfect partitions of the number p so that a generating function when expanded is

$$A_1 x + A_2 x^2 + A_3 x^3 + \ldots + A_p x^p + \ldots .$$

The perfect partitions having a highest part equal to $p+1$ are derivable from the perfect partitions of the number p by prefixing to each of the latter the part $p+1$ any number of times repeated. Hence, the perfect partitions, of all numbers, which have a highest part equal to $p+1$ are given in number by the coefficients of x in the expansion of the fraction

$$\frac{A_p x^{2p+1}}{1 - x^{p+1}}.$$

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Hence the algebraical identity

$$\frac{A_0 x}{1-x} + \frac{A_1 x^2}{1-x^2} + \frac{A_2 x^3}{1-x^3} + \ldots + \frac{A_p x^{2p+1}}{1-x^{p+1}} + \ldots,$$

$$= A_1 x + A_2 x^2 + A_3 x^3 + \ldots + A_p x^p + \ldots + A_{2p+1} x^{2p+1} + \ldots .$$

The arithmetical interpretation is the relation

$$A_{2p+1} = \Sigma A_s,$$

the summation being for all values of s which make $s+1$ a divisor of $p+1$.

For example,

$$A_{11} = A_0 + A_1 + A_2 + A_5,$$

since $\frac{1}{2}(11+1) = 6$ has the divisons 1, 2, 3, and 6.

The theorem may be written

$$A_{2p-1} = \Sigma A_{s-1},$$

where s is a divisor of p and the summation is for all such.

11. Instead of viewing the aggregate of perfect partitions of all numbers in respect of the highest part involved, it is interesting to regard the equation from the point of view of the number of parts involved, for herein lies another connexion with the theory of compositions.

It can be shewn that there is a one-to-one correspondence between the perfect partitions containing s parts and the compositions of the number s.

Write down the perfect partitions of less than five parts as follows:—

$$(1)$$

$$(1^2) \quad , \quad (21)$$

$$(1^3) \quad , \quad (31^2) \quad , \quad (2^21) \quad , \quad (421)$$

$$(1^4) \ , \ (41^3) \ , \ (3^21^2) \ , \ (631^2) \ , \ (2^31) \ , \ (62^21) \ , \ (4^221) \ , \ (8421).$$

It will be observed that in any line each perfect partition gives birth to two perfect partitions in the line below

(*a*) by prefixing a part equal to the existing highest part.

(*b*) by prefixing a part which is greater by unity than the sum of all the existing parts.

Write down now, instead of the perfect partitions themselves, the repetitional numbers of the parts of those partitions.

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We thus obtain a scheme:—

1

2 , 11

3 , 12 , 21 , 111

4 , 13 , 22 , 112 , 31 , 121 , 211 , 1111;

and the 1st, 2nd, 3rd and 4th lines are seen to represent the compositions of the numbers 1, 2, 3 and 4.

Each composition in a line gives rise to two compositions in the line below by rules derived from (a) and (b). These are:—

(a') increase the first part by unity,

(b') prefix a part unity.

These are in fact rules for deriving the compositions of a number from those of the next preceding number. That the rules are correct is clear, because no two compositions thus obtained can be identical, and, further, the number of compositions is right, viz. 2^{s-1} in the case of a number s.

Hence the theorem

"There is a one-to-one correspondence between the perfect partitions containing s parts and the compositions of the number S."

The number, as seen, is 2^{s-1}, and the correspondence is in reference to the repetitional numbers occurring in the perfect partitions.*

It is remarkable that the correspondence in this case should be with compositions of simple or unipartite numbers.

The main theorem of the paper exhibits a correspondence with the compositions of multipartite numbers in general, and I believe marks the first appearance of such compositions in the theory of Numbers. This, I think, shews that the subject of perfect partitions is one worthy of the attention of mathematicians.

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* It will be readily seen that the general form of a perfect partition is

$$1^{\alpha}(1+\alpha)^{\beta}(1+\alpha.1+\beta)^{\gamma}(1+\alpha.1+\beta.1+\gamma)^{\delta}\ldots,$$

where $\alpha, \beta, \gamma, \delta, \ldots$, are any positive, non-zero integers taken at pleasure. There is thus one, and only one, perfect partition in correspondence with every succession of positive integers

$$\alpha, \beta, \gamma, \delta, \ldots .$$

The aggregate of these successions constitutes the aggregate of the compositions for all (unipartite) numbers.

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CORRECTION TO THE NOTE ON SCHLÄFLI'S MODULAR EQUATION FOR THE CUBIC TRANSFORMATION, pp. 59–60.

By Prof. *Cayley*.

I FIND that I misquoted Schläfli's equation, viz. in effect I took it to be

$$S_1^4 + T_1^4 + 8S_1T_1 - S_1^3T_1^3 = 0; \text{ where } S_1 = \frac{\sqrt{2}}{\sqrt[4]{(kk')}}, \; T_1 = \frac{\sqrt{2}}{\sqrt[4]{(\lambda\lambda')}},$$

whereas his equation really is

$$S^4 + T^4 - 8ST + S^3T^3 = 0; \text{ where } S = 2\sqrt[4]{(kk')}, \; T = 2\sqrt[4]{(\lambda\lambda')}.$$

The change is only a change of form, for writing $S_1 = \frac{2\sqrt{2}}{S}$ and $T_1 = \frac{2\sqrt{2}}{T}$, the equation in (S_1, T_1) is converted into that in (S, T); but it was quite an unnecessary one, and I cannot account for having made it, as the paper in *Crelle* must have been before me.

THEOREM RELATING TO SUMS OF EVEN POWERS OF THE NATURAL NUMBERS.

By *J. W. L. Glaisher*.

Statement of the Theorem, §§ 1–4.

§1. LET S_n denote the series

$$1^n + 2^n + 3^n + \ldots + i^n;$$

then it can be shown that

$$S_2 = \tfrac{1}{6} i (i+1)(2i+1),$$

$$S_4 = \tfrac{1}{30} i (i+1)(2i+1)(3i^2+3i-1),$$

$$S_6 = \tfrac{1}{42} i (i+1)(2i+1)\{3(i^2+i)^2 - 3(i^2+i) + 1\}.$$

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Now $S_1 = \frac{1}{2}i(i+1)$, so that

$$\frac{S_2}{S_1} = \tfrac{1}{3}(2i+1),$$

$$\frac{S_4}{S_1} = \tfrac{1}{15}(2i+1)(3i^2+3i-1),$$

$$\frac{S_6}{S_1} = \tfrac{1}{21}(2i+1)\{3(i^2+i)^2-3(i^2+i)+1\}.$$

From these equations we readily deduce the formulæ

$$\frac{S_2}{S_1} = \tfrac{1}{3}(2i+1) = \tfrac{2}{3}(S_0+\tfrac{1}{2}),$$

$$\frac{S_4}{S_1} = \tfrac{6}{5}S_2 - \tfrac{2}{15}(S_0+\tfrac{1}{2}),$$

$$\frac{S_6}{S_1} = \tfrac{10}{7}S_4 - \tfrac{4}{7}S_2 + \tfrac{2}{21}(S_0+\tfrac{1}{2}).$$

§ 2. In general, it can be shown that

$$S_{2n} = i(i+1)(2i+1)F(i^2+i),$$

where $F(i^2+i)$ is a rational and integral function of i^2+i of the degree $n-1$. It is therefore evident that we may express S_{2n} as a linear function of S_{2n-2}, S_{2n-4}, ..., S_2 and $S_0+\frac{1}{2}$. I have had occasion to determine the coefficients in this expansion of S_{2n}, and find that they admit of being expressed in a simple and elegant form in terms Bernoullian numbers. The result may be stated as follows:

$$\text{If } \frac{1^{2n}+2^{2n}+3^{2n}+\ldots+i^{2n}}{1+2+3+\ldots+i} = A_2(1^{2n-2}+2^{2n-2}+3^{2n-2}+\ldots+i^{2n-2})$$
$$+ A_4(1^{2n-4}+2^{2n-4}+3^{2n-4}+\ldots+i^{2n-4}),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$+ A_{2n-2}(1^2+2^2+3^2+\ldots+i^2),$$
$$+ A_{2n}(i+\tfrac{1}{2}),$$

then

$$\tfrac{1}{2}A_2 = \frac{2n-1}{2n+1},$$

$$\tfrac{1}{2}A_4 = \frac{2n-1}{2n+1} - 2nB_1,$$

$$\tfrac{1}{2}A_6 = \frac{2n-1}{2n+1} - 2nB_1 + \frac{2n(2n-1)(2n-2)}{3!}\frac{B_2}{2},$$

$$\tfrac{1}{2}A_8 = \frac{2n-1}{2n+1} - 2nB_1 + \frac{2n(2n-1)(2n-2)}{3!}\frac{B_2}{2} - \frac{2n(2n-1)\ldots(2n-4)}{5!}\frac{B_3}{3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where B_1, B_2, B_3, ... are the Bernoullian numbers.

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§ 3. Using the notation $n^{(r)}$ to represent the factorial $n(n-1)\ldots(n-r+1)$, we may write these equations in the form

$$\tfrac{1}{2}A_2 = \frac{2n-1}{2n+1},$$

$$\tfrac{1}{2}A_4 = \frac{2n-1}{2n+1} - 2nB_1,$$

$$\tfrac{1}{2}A_6 = \frac{2n-1}{2n+1} - 2nB_1 + \frac{(2n)^{(3)}}{3!}\frac{B_2}{2},$$

$$\tfrac{1}{2}A_8 = \frac{2n-1}{2n+1} - 2nB_1 + \frac{(2n)^{(3)}}{3!}\frac{B_2}{2} - \frac{(2n)^{(5)}}{5!}\frac{B_3}{3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or in the form

$$\tfrac{1}{2}(2n+1)A_2 = \tfrac{1}{2}(2n-1),$$

$$\tfrac{1}{2}(2n+1)A_4 = \tfrac{1}{2}(2n-1) - \frac{(2n+1)^{(2)}}{2!}B_1,$$

$$\tfrac{1}{2}(2n+1)A_6 = \tfrac{1}{2}(2n-1) - \frac{(2n+1)^{(2)}}{2!}B_1 + \frac{(2n+1)^{(4)}}{4!}B_2,$$

$$\tfrac{1}{2}(2n+1)A_8 = \tfrac{1}{2}(2n-1) - \frac{(2n+1)^{(2)}}{2!}B_1 + \frac{(2n+1)^{(4)}}{4!}B_2 - \frac{(2n+1)^{(6)}}{6!}B_3,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

§ 4. The calculation of the successive coefficients $A_2, A_4, \ldots$ may be readily effected by means of the formulæ

$$\tfrac{1}{2}A_2 = \frac{2n-1}{2n+1},$$

$$\tfrac{1}{2}A_4 = \tfrac{1}{2}A_2 - 2nB_1,$$

$$\tfrac{1}{2}A_6 = \tfrac{1}{2}A_4 - \frac{(2n)^{(3)}}{3!}\frac{B_2}{2},$$

$$\tfrac{1}{2}A_8 = \tfrac{1}{2}A_6 - \frac{(2n)^{(5)}}{5!}\frac{B_3}{3},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,$$

As examples, putting $n = 2$,

$$\tfrac{1}{2}A_2 = \tfrac{3}{5}; \quad \tfrac{1}{2}A_4 = \tfrac{3}{5} - \tfrac{4}{6} = -\tfrac{1}{15};$$

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putting $n=4$,

$$\tfrac{1}{2}A_2=\tfrac{5}{7},\ \tfrac{1}{2}A_6=\tfrac{5}{7}-\tfrac{8}{6}=-\tfrac{2}{7},\ \tfrac{1}{2}A_8=-\tfrac{2}{7}+\frac{6.5.4}{2.3}\cdot\tfrac{1}{6\,0}=\tfrac{1}{2\,1};$$

putting $n=6$,

$$\tfrac{1}{2}A_2=\tfrac{7}{9},\ \tfrac{1}{2}A_4=\tfrac{7}{9}-\tfrac{8}{6}=-\tfrac{8}{9},\ \tfrac{1}{2}A_6=-\tfrac{5}{9}+\frac{8.7.6}{2.3}\cdot\tfrac{1}{6\,0}=\tfrac{1}{4\,5},$$

$$\tfrac{1}{2}A_8=\tfrac{1}{4\,5}-\frac{8.7.6.5.4}{2.3.4.5}\cdot\tfrac{1}{1\,2\,6}=-\tfrac{1}{1\,5}.$$

Proof of the Theorem, §§ 5–10.

§ 5. The theorem may be proved in the following manner. We have

$$\frac{S_{2n}}{S_1}=A_2S_{2n-2}+A_4S_{2n-4}+\ldots+A_{2n-2}S_2+A_{2n}(S_0+\tfrac{1}{2}),$$

and we know that

$$S_n=\frac{i^{n+1}}{n+1}+\tfrac{1}{2}i^n+\frac{B_1}{2}\frac{n}{1\,!}i^{n-1}-\frac{B_2}{4}\frac{n^{(3)}}{3\,!}i^{n-3}+\frac{B_3}{6}\frac{n^{(5)}}{5\,!}i^{n-5}-\&c.$$

The coefficients A_2, A_4, ..., A_{2n} may therefore be determined by equating coefficients in the equation

$$\frac{i^{2n+1}}{2n+1}+\tfrac{1}{2}i^{2n}+\frac{B_1}{2}\frac{(2n)^{(1)}}{1\,!}i^{2n-1}-\frac{B_2}{4}\frac{(2n)^{(3)}}{3\,!}i^{2n-3}+\frac{B_3}{6}\frac{(2n)^{(5)}}{5\,!}i^{2n-5}-\&c.$$

$$=\tfrac{1}{2}(i^2+i)\times$$

$$\Big[A_2\Big\{\frac{i^{2n-1}}{2n-1}+\tfrac{1}{2}i^{2n-2}+\frac{B_1}{2}\frac{(2n-2)^{(1)}}{1\,!}i^{2n-3}-\frac{B_2}{4}\frac{(2n-2)^{(3)}}{3\,!}i^{2n-5}+\&c.\Big\}$$

$$+A_4\Big\{\frac{i^{2n-3}}{2n-3}+\tfrac{1}{2}i^{2n-4}+\frac{B_1}{2}\frac{(2n-4)^{(1)}}{1\,!}i^{2n-5}-\frac{B_2}{4}\frac{(2n-4)^{(3)}}{3\,!}i^{2n-7}-\&c.\Big\}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\Big].$$

§ 6. Now put $2n-1=m$, and let

$$\tfrac{1}{2}A_2=ma_1,\ \tfrac{1}{2}A_4=\frac{m^{(3)}}{3\,!}a_3,\ \tfrac{1}{2}A_6=\frac{m^{(5)}}{5\,!}a_5,\ \ldots,$$

$$\tfrac{1}{2}A_r=\frac{m^{(2r-1)}}{(2r-1)!}a_{2r-1}=\frac{m^{(2n-2r+1)}}{(2n-2r+1)!}a_{2n-1},\ \ldots,\ \tfrac{1}{2}A_{2n}=\frac{m^{(2n-1)}}{(2n-1)!}a_m=\frac{m^{(3)}}{2\,!}a_m,$$

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The equality then becomes

$$\frac{2i^{m+2}}{m+2}+i^{m+1}+\frac{B_1}{1}(m+1)\,i^m-\frac{B_2}{2}\frac{(m+1)^{(3)}}{3!}i^{m-1}+\&c.$$

$$=(i^2+i)\Big\{2a_1i^m+a_1mi^{m-1}+a_1\frac{B_1}{1}\frac{m^{(2)}}{1!1!}i^{m-2}-a_1\frac{B_2}{2}\frac{m^{(4)}}{1!3!}i^{m-4}+\&c.$$

$$+2a_3\frac{m^{(2)}}{3!}i^{m-2}+a_3\frac{m^{(3)}}{3!}i^{m-3}+a_3\frac{B_1}{1}\frac{m^{(4)}}{3!1!}i^{m-4}-a_3\frac{B_2}{2}\frac{m^{(6)}}{3!3!}i^{m-6}+\&c.$$

$$+2a_5\frac{m^{(4)}}{5!}i^{m-4}+a_5\frac{m^{(5)}}{5!}i^{m-5}+a_5\frac{B_1}{1}\frac{m^{(6)}}{5!1!}i^{m-6}-a_5\frac{B_2}{2}\frac{m^{(8)}}{5!3!}i^{m-8}+\&c.$$

$$\cdots\cdots\Big\}.$$

§ 7. Equating the even powers, we have

$$i^{m+1}$$

$$=a_1mi^{m+1}\quad+\quad 2a_1i^{m+1}+a_1\frac{B_1}{1}\frac{m^{(2)}}{1!1!}i^{m-1}-a_1\frac{B_2}{2}\frac{m^{(4)}}{1!3!}i^{m-3}+\&c.$$

$$+a_3\frac{m^{(3)}}{3!}i^{m-1}+2a_3\frac{m^{(2)}}{3!}i^{m-1}+a_3\frac{B_1}{1}\frac{m^{(4)}}{3!1!}i^{m-3}-a_3\frac{B_2}{2}\frac{m^{(6)}}{3!3!}i^{m-5}+\&c.$$

$$+a_5\frac{m^{(5)}}{5!}i^{m-3}+2a_5\frac{m^{(4)}}{5!}i^{m-3}+a_5\frac{B_1}{1}\frac{m^{(6)}}{5!1!}i^{m-5}-a_5\frac{B_2}{2}\frac{m^{(8)}}{5!3!}i^{m-7}+\&c.$$

$$\cdots\cdots,$$

whence, by equating the coefficients of i^{m+1}, i^{m-1}, i^{m-3}, &c.,

$$2a_1+ma_1=1,$$

$$\frac{m^{(3)}}{3!}a_3+2\frac{m^{(2)}}{3!}a_3+\frac{m^{(2)}}{1!1!}\frac{B_1}{1}a_1=0,$$

$$\frac{m^{(5)}}{5!}a_5+2\frac{m^{(4)}}{5!}a_5+\frac{m^{(4)}}{3!1!}\frac{B_1}{1}a_3-\frac{m^{(4)}}{1!3!}\frac{B_2}{2}a_1=0,$$

$$\frac{m^{(7)}}{7!}a_7+2\frac{m^{(6)}}{7!}a_7+\frac{m^{(6)}}{5!1!}\frac{B_1}{1}a_5-\frac{m^{(6)}}{3!3!}\frac{B_2}{2}a_3+\frac{m^{(6)}}{1!5!}\frac{B_3}{3}a_1=0,$$

$$\cdots\cdots,$$

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§ 8. Equating the uneven powers in § 6, we have

$$\frac{2i^{m+2}}{m+2} + \frac{B_1}{1}(m+1)\,i^m - \frac{B_2}{2}\frac{(m+1)^{(3)}}{3!}\,i^{m-2} + \&c.$$

$$= 2a_1 i^{m+2} + a_1 m i^m + a_1 \frac{B_1}{1}\frac{m^{(2)}}{1!\,1!}\,i^m - a_1\frac{B_2}{2}\frac{m^{(4)}}{1!\,3!}\,i^{m-2} + \&c.$$

$$+ 2a_3\frac{m^{(2)}}{3!}\,i^m + a_3\frac{m^{(3)}}{3!}\,i^{m-2} + a_3\frac{B_1}{1}\frac{m^{(4)}}{3!\,1!}\,i^{m-2} - a_3\frac{B_2}{2}\frac{m^{(6)}}{3!\,3!}\,i^{m-4} + \&c.$$

$$+ 2a_5\frac{m^{(4)}}{5!}\,i^{m-2} + a_5\frac{m^{(5)}}{5!}\,i^{m-4} + a_5\frac{B_1}{1}\frac{m^{(6)}}{5!\,1!}\,i^{m-4} - a_5\frac{B_2}{2}\frac{m^{(8)}}{5!\,3!}\,i^{m-6} + \&c.$$

.....,

whence, by equating the coefficients of i^{m+2}, i^m, i^{m-2}, &c.,

$$2a_1 = \frac{2}{m+2},$$

$$ma_1 + 2\frac{m^{(3)}}{3!}a_3 + \frac{m^{(2)}}{1!\,1!}\frac{B_1}{1}a_1 = (m+1)\frac{B_1}{1},$$

$$\frac{m^{(3)}}{3!}a_3 + 2\frac{m^{(4)}}{5!}a_5 + \frac{m^{(4)}}{3!\,1!}\frac{B_1}{1}a_3 - \frac{m^{(4)}}{1!\,3!}\frac{B_2}{2}a_1 = -\frac{(m+1)^{(3)}}{3!}\frac{B_2}{2},$$

$$\frac{m^{(5)}}{5!}a_5 + 2\frac{m^{(6)}}{7!}a_7 + \frac{m^{(6)}}{5!\,1!}\frac{B_1}{1}a_5 - \frac{m^{(6)}}{3!\,3!}\frac{B_2}{2}a_3 + \frac{m^{(6)}}{1!\,5!}\frac{B_3}{3}a_1 = \frac{(m+1)^{(5)}}{5!}\frac{B_3}{3},$$

.....

§ 9. By subtracting these equations from those in § 7, we find

$$ma_1 = 1 - \frac{2}{m+2},$$

$$\frac{m^{(3)}}{3!}a_3 - ma_1 = -(m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5!}a_5 - \frac{m^{(3)}}{3!}a_3 = \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7!}a_7 - \frac{m^{(5)}}{5!}a_5 = -\frac{(m+1)^{(5)}}{5!}\frac{B_3}{3},$$

&c., &c.,

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giving

$$a_1 = \frac{1}{m+2},$$

$$\frac{m^{(3)}}{3!}a_3 = \frac{m}{m+2} - (m+1)\frac{B_1}{1},$$

$$\frac{m^{(5)}}{5!}a_5 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2},$$

$$\frac{m^{(7)}}{7!}a_7 = \frac{m}{m+2} - (m+1)\frac{B_1}{1} + \frac{(m+1)^{(3)}}{3!}\frac{B_2}{2} - \frac{(m+1)^{(5)}}{5!}\frac{B_3}{3}.$$

&c. &c.

§ 10. By substituting for ma_1, $\frac{m^{(3)}}{3!}a_3$, ... their values $\frac{1}{2}A_2$, $\frac{1}{2}A_4$, ... (§ 6) and replacing m by $2n-1$, these relations become

$$\frac{1}{2}A_2 = \frac{2n-1}{2n+1},$$

$$\frac{1}{2}A_4 - \frac{1}{2}A_2 = -2n\frac{B_1}{1},$$

$$\frac{1}{2}A_6 - \frac{1}{2}A_4 = \frac{(2n)^{(3)}}{3!}\frac{B_2}{2},$$

&c., &c.,

and

$$\frac{1}{2}A_2 = \frac{2n-1}{2n+1},$$

$$\frac{1}{2}A_4 = \frac{2n-1}{2n+1} - 2n\frac{B_1}{1},$$

$$\frac{1}{2}A_6 = \frac{2n-1}{2n+1} - 2n\frac{B_1}{1} + \frac{(2n)^{(3)}}{3!}\frac{B_2}{2},$$

&c., ... &c.,

which are the results stated in §§ 3 and 4.

The quantities $a_1, a_3, \ldots$, §§ 11, 12.

§ 11. The quantities $a_1, a_3, \ldots, a_m$ are the coefficients in the equation

$$S_{m+1} = 2(i^2+i)\{ma_1 S_{m-1} + \frac{m^{(3)}}{3!}a_3 S_{m-3} + \frac{m^{(5)}}{5!}a_5 S_{m-5} + \ldots + \frac{m^{(3)}}{2!}a_{m-2}S_2 + a_m(S_0 + \tfrac{1}{2})\},$$

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where m is any uneven number. Writing the S's at full length and putting $1+2+3+\ldots+i$ in place of $\frac{1}{2}(i^2+i)$, this equation becomes

$$\frac{1}{4}\frac{1^{m+1}+2^{m+1}+3^{m+1}+\ldots+i^{m+1}}{1+2+3+\ldots+i} = ma_1(1^{m-1}+2^{m-1}+3^{m-1}+\ldots+i^{m-1})$$

$$+\frac{m^{(3)}}{3!}a_3(1^{m-3}+2^{m-3}+3^{m-3}+\ldots+i^{m-3}),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{m^{(3)}}{2!}a^{m-2}(1^2+2^2+3^2+\ldots+i^2)$$

$$+a_m(i+\tfrac{1}{2}).$$

It was in fact in order to determine the values of $a_1, a_3, \ldots, a_m$ in this equation that I was led to obtain the solution of the question to which this paper relates.* There was of course no actual occasion in § 6 to replace the quantities $A_3, A_4, \ldots$ by $a_1, a_3, \ldots$, but having worked out the question originally by means of the latter notation, I found that the equations suffered a loss of simplicity and regularity by the employment of $A_3, A_4, \ldots$ instead of $a_1, a_3, \ldots$.

§ 12. The equations in § 10, give

$$a_1 = \frac{1}{m+2},$$

$$a_3 = -\frac{1}{(m+2)\,m},$$

$$a_5 = \frac{m+10}{3\,(m+2)\,m\,(m-2)},$$

$$a_7 = -\frac{m^2+12m+56}{3\,(m+2)\,m\,(m-2)\,(m-4)},$$

&c., &c.,

* See the next paper "Recurring relations involving sums of powers of divisons" (§ 7), p. 129.

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from which we deduce for A_2, A_4, ..., the values

$$A_2 = 2\frac{2n-1}{2n+1},$$

$$A_4 = -\frac{2(n-1)(2n-3)}{3(2n+1)},$$

$$A_6 = \frac{(n-1)(n-2)(2n-5)(2n+9)}{45(2n+1)},$$

$$A_8 = -\frac{(n-1)(n-2)(n-3)(2n-7)(4n^2+20n+45)}{945(2n+1)}.$$

&c. &c.

§ 13. The relations connecting a_1, a_3, ... in § 7 seem to be deserving of notice. It will be observed that $m^{(2)}$, $m^{(4)}$, ... divide out from the second, third, ... equations, leaving

$$a_1(m+2) = 1,$$

$$\frac{m}{3}a_3 + 2\frac{B_1}{1}a_1 = 0,$$

$$\frac{m-2}{5}a_5 + 4\frac{B_1}{1}a_3 - 4\frac{B_2}{2}a_1 = 0,$$

$$\frac{m-4}{7}a_7 + 6\frac{B_1}{1}a_5 - \frac{6^{(3)}}{3!}\frac{B_2}{2}a_3 + 6\frac{B_3}{3}a_1 = 0,$$

$$\frac{m-6}{9}a_9 + 8\frac{B_1}{1}a_7 - \frac{8^{(3)}}{3!}\frac{B_2}{2}a_5 + \frac{8^{(5)}}{3!}\frac{B_3}{3}a_3 - 8\frac{B_4}{4}a_1 = 0.$$

.....

It was from these equations that I originally calculated the values of a_1, a_3 ... given in the preceding section, before it had occured to me to write down also the system of equations derived from the uneven powers (§ 8). It then became apparent that much simpler relations, connecting together two consecutive a's, could be obtained by subtraction, and that by their means the general value of a_{2r-1} could be at once written down (§ 9). These equations are, however, of some interest, as affording a simple verification of the values of a_1, a_3, ... when calculated.

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RECURRING RELATIONS INVOLVING SUMS OF POWERS OF DIVISORS.

(*Second Paper*).

By *J. W. L. Glaisher.*

§ 1. The theorem given on p. 130, which connects together the sums of the uneven powers of the divisors of the numbers n, $n-1$, $n-3$, $n-6$, &c. may be written:

$$\sigma_m(n) - 3\sigma_m(n-1) + 5\sigma_m(n-3) - 7\sigma_m(n-6) + \&c.$$

$$= 2\frac{m^{(2)}}{2!}\{\sigma_{m-2}(n-1) - (1^2+2^2)\sigma_{m-2}(n-3) + (1^2+2^2+3^2)\sigma_{m-2}(n-6) - \&c.\}$$

$$= 2\frac{m^{(4)}}{4!}\{\sigma_{m-4}(n-1) - (1^4+2^4)\sigma_{m-4}(n-3) + (1^4+2^4+3^4)\sigma_{m-4}(n-6) - \&c.\}$$

.....

$$- 2m\{\sigma(n-1) - (1^{m-1}+2^{m-1})\sigma(n-3) + (1^{m-1}+2^{m-1}+3^{m-1})\sigma(n-6) - \&c.\}$$

$$+ [(-1)^{\sigma-1}(1^{m+1}+2^{m+1}+3^{m+1}+\ldots+g^{m+1})],$$

where m is any uneven number, $m^{(s)}$ denotes the factorial $m(m-1)\ldots(m-s+1)$, and $\sigma_r(n)$ denotes the sum of the r^{th} powers of the divisors of n. The additional term in square brackets is to be included when n is a triangular number $\frac{1}{2}g(g+1)$.

§ 2. I have obtained another general theorem of the same class, in which the coefficients are simpler and the different series have a closer analogy to the original formula

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6)\ \&c. = [(-1)^{\sigma-1}(1^2+2^2+3^2+\ldots+g^2],$$

of which the above theorem is a generalisation.

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Using the same notation as in § 1, this theorem may be written:

$$\sigma_m(n) - 3\sigma_m(n-1) + 5\sigma_m(n-3) - 7\sigma_m(n-6) + \&c.$$

$$+ \frac{m^{(3)}}{2^2.3!}\{\sigma_{m-2}(n) - 3^3\sigma_{m-2}(n-1) + 5^3\sigma_{m-2}(n-3) - 7^3\sigma_{m-2}(n-6) + \&c.\}$$

$$+ \frac{m^{(4)}}{2^4.5!}\{\sigma_{m-4}(n) - 3^5\sigma_{m-4}(n-1) + 5^5\sigma_{m-4}(n-3) - 7^5\sigma_{m-4}(n-6) + \&c.\}$$

.....

$$+ \frac{m^{(3)}}{2^{m-3}.3!}\{\sigma_3(n) - 3^{m-3}\sigma_3(n-1) + 5^{m-3}\sigma_3(n-3) - 7^{m-3}\sigma_3(n-6) + \&c.\}$$

$$+ \frac{1}{2^{m-1}}\{\sigma(n) - 3^m\sigma(n-1) + 5^m\sigma(n-3) - 7^m\sigma(n-6) + \&c.\}$$

$$= [(-1)^{g-1}\frac{1}{2^{m+1}(m+1)}\{g(2g+1)^{m+1} - 1^{m+1} - 3^{m+1} - 5^{m+1} - \ldots - (2g-1)^{m+1}\}].$$

§ 3. As in the case of the former theorem (pp. 132–134), we may dispense with the additional term by assigning to $\sigma(0)$, $\sigma_3(0)$, ... values proportional to n.

Thus, for example, putting $m = 3$, we find

$$\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.$$

$$+ \tfrac{1}{4}\{\sigma(n) - 3^3\sigma(n-1) + 5^3\sigma(n-3) - 7^3\sigma(n-6) + \&c.\}$$

$$= [(-1)^{g-1}\frac{(2g+1)(g^2+g)(12g^2+12g+1)}{120}],$$

and we may dispense with the additional term, if we put

$$\sigma(0) = \tfrac{1}{6}n, \quad \sigma_3(0) = \tfrac{1}{30}n.$$

For $m = 5$, we find

$$\sigma_5(n) - 3\sigma_5(n-1) + 5\sigma_5(n-3) - 7\sigma_5(n-6) + \&c.$$

$$+ \tfrac{5}{6}\{\sigma_3(n) - 3^3\sigma_3(n-1) + 5^3\sigma_3(n-3) - 7^3\sigma_3(n-6) + \&c.\}$$

$$+ \tfrac{1}{12}\{\sigma(n) - 3^5\sigma(n-1) + 5^5\sigma(n-3) - 7^5\sigma(n-6) + \&c.\}$$

$$= [(-1)^{g-1}\frac{(2g+1)(g^2+g)\{144(g^2+g)^2 + 24(g^2+g) + 13\}]}{2016},$$

and we may dispense with the additional term by putting

$$\sigma(0) = \tfrac{1}{7}n, \quad \sigma_3(0) = -\tfrac{1}{70}n, \quad \sigma_5(0) = \tfrac{1}{63}n,$$

and so on for higher values of m.

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§ 4. A very interesting result in connexion with the general formula in § 2, is that it is possible to assign to $\sigma(0)$, $\sigma_3(0)$, ... a system of constant values (*i.e.* values independent of *both* m and n) by means of which the additional term can be so nearly represented that it is reduced to a single monomial expression.

These values are

$$\sigma(0) = -\tfrac{1}{4}B_1,\ \sigma_3(0) = \tfrac{1}{4}\frac{B_2}{2},\ \sigma_5(0) = -\tfrac{1}{4}\frac{B_3}{3},\ \ldots,$$

and in general

$$\sigma_{2r-1}(0) = (-1)^r \tfrac{1}{4}\frac{B_r}{r},$$

B_1, B_2, B_3, ... being the Bernoullian numbers.

When these values are assigned to $\sigma(0)$, $\sigma_3(0)$, ... the additional term in the general formula becomes

$$\left[(-1)^{\sigma-1}\frac{(2g+1)^{m+2}}{2^{m+2}(m+2)}\right].$$

§ 5. Putting f for $2g+1$, so that f is the coefficient of the term of zero argument (when it occurs) in the first series, the general formula becomes, when $m=1$,

$$\sigma(n) - 3\sigma(n-1) + 5\sigma(n-3) - 7\sigma(n-6) + \&c. = \left[(-1)^{\frac{1}{2}(f^2+1)}\frac{f^3}{24}\right],$$

where $\sigma(0) = -\frac{1}{24}$.

For example, putting $n=6$,

$$\sigma(6) - 3\sigma(5) + 5\sigma(3) - 7\sigma(0) = \frac{7^3}{24},$$

that is $1+2+3+6-3(1+5)+5(1+3)+\frac{7}{24} = \frac{7^3}{24}$.

For $m=3$, we have

$$\sigma_3(n) - 3\sigma_3(n-1) + 5\sigma_3(n-3) - 7\sigma_3(n-6) + \&c.$$
$$+ \tfrac{1}{4}\{\sigma(n) - 3^5\sigma(n-1) + 5^5\sigma(n-3) - 7^5\sigma(n-6) + \&c.\}$$
$$= \left[(-1)^{\frac{1}{2}(f^2+1)}\frac{f^5}{160}\right],$$

where $\sigma(0) = -\frac{1}{24}$, $\sigma_3(0) = \frac{1}{24}\sigma$.

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Putting $n=6$, the formula gives

$$1^5+2^5+3^5+6^5-3\ (1^5+5^5)+5\ (1^5+3^5)-\frac{7}{240}$$
$$+\tfrac{1}{4}\left\{1+2+3+6-3^5(1+5)+5^5(1+3)+\frac{7^5}{24}\right\}=\frac{7^5}{160}.$$

For $m=5$, we have

$$\sigma_5(n)-3\sigma_5\ (n-1)+5\sigma_5\ (n-3)-7\sigma_5\ (n-6)+\&c.$$
$$+\tfrac{5}{6}\ \{\sigma_3(n)-3^3\sigma_3(n-1)+5^3\sigma_3(n-3)-7^3\sigma_3(n-6)+\&c.\}$$
$$+\tfrac{1}{16}\{\sigma(n)-3^5\sigma\ (n-1)+5^5\sigma\ (n-3)-7^5\sigma\ (n-6)+\&c.\}$$
$$=\left[(-1)^{\frac{1}{2}(f^2+1)}\frac{f^7}{896}\right],$$

where $\sigma(0)=-\frac{1}{24}$, $\sigma_3(0)=\frac{1}{240}$, $\sigma_5(0)=-\frac{1}{504}$.

Putting $n=6$,

$$1^5+2^5+3^5+6^5-3\ (1^5+5^5)+5\ (1^5+3^5)+\frac{7}{504}$$
$$+\tfrac{5}{6}\left\{1^5+2^5+3^5+6^5-3^3(1^5+5^5)+5^3(1^5+3^5)-\frac{7^3}{240}\right\}$$
$$+\tfrac{1}{16}\left\{1+2+3+6-3^5(1+5)+5^5(1+3)+\frac{7^5}{24}\right\}=\frac{7^7}{896}.$$

§ 6. It will be observed that we can dispense with the additional term entirely by making *one* of the quantities $\sigma(0)$, $\sigma_3(0)$, ... depend upon m and f, the others remaining constant. For example, the additional term disappears, if we put

$$\sigma(0)=\tfrac{1}{8}\left\{\frac{f^2}{m+2}-\tfrac{1}{8}\right\},$$

the values of $\sigma_3(0)$, $\sigma_5(0)$, ... being $\frac{1}{240}$, $-\frac{1}{504}$, ..., as before.

Since $f^2-1=(g^2+g)=8n$, we see that this value of $\sigma(0)$ may be written

$$\sigma(0)=\frac{n}{m+2}-\tfrac{1}{24}\frac{m-1}{m+2}.$$

Thus, for $m=1$, $\sigma(0)=\frac{1}{3}n$,

" $m=3$, $\sigma(0)=\frac{1}{5}n-\frac{1}{10}$,

" $m=5$, $\sigma(0)=\frac{1}{7}n-\frac{1}{12}$,

" $m=7$, $\sigma(0)=\frac{1}{9}n-\frac{1}{36}$,

&c., &c.

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§ 7. The method of representing the additional term given in § 4 may be established as follows.

We know that*, m being uneven,

$$1^{m+1}+3^{m+1}+5^{m+1}+\ldots+f^{m+1}$$

$$=\tfrac{1}{2}\frac{f^{m+2}}{m+2}+\tfrac{1}{2}f^{m+1}+(m+1)^{(1)}B_1 f^m-2^3\frac{(m+1)^{(3)}}{3!}\frac{B_2}{2}f^{m-2}+2^4\frac{(m+1)^{(5)}}{5!}\frac{B_3}{3}f^{m-4}$$

$$-\ldots\pm 2^{m-1}(m+1)^{(1)}\frac{B^{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}f.$$

Now the additional term in § 2, putting f for $2g+1$,

$$=(-1)^{\frac{1}{2}(f+1)}\frac{1}{2^{m+1}(m+1)}\left\{\frac{(f+1)f^{m+1}}{2}-1^{m+1}-3^{m+1}-5^{m+1}-\ldots-f^{m+1}\right\}$$

$$=(-1)^{\frac{1}{2}(f+1)}\left\{\frac{f^{m+2}}{2^{m+2}(m+2)}-B_1\frac{f^m}{2^{m+1}}+\frac{m^{(3)}}{3!}\frac{B_2}{2}\frac{f^{m+2}}{2^{m-2}}\right.$$

$$\left.-\frac{m^{(4)}}{5!}\frac{B_3}{3}\frac{f^{m-4}}{2^{m-3}}-\ldots\pm\frac{B_{\frac{1}{2}(m+1)}}{\frac{1}{2}(m+1)}\frac{f}{2^2}\right\},$$

and by comparing this expression with the left-hand member of the formula in § 2, it is evident that we may represent all the terms after the first by putting

$$\sigma(0)=-\tfrac{1}{4}B_1,\ \sigma_2(0)=\tfrac{1}{4}\frac{B_2}{2},\ \sigma_5(0)=-\tfrac{1}{4}\frac{B_3}{3},\ \&c.$$

When this is done the first term, viz.

$$\left[(-1)^{\frac{1}{2}(f+1)}\frac{f^{m+2}}{2^{m+2}(m+2)}\right],$$

is the only one left remaining on the right-hand side of the equation.

The proof of the theorem in § 2 will be included in the paper referred to on p. 135.

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* This was proved on p. 173.

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SOME ARITHMETICAL FORMULAE.

By *J. Hammond.*

(Continued from p. 168*).*

6. LET $P(m, n)$

$$= \{1 - (m:2)\,(n:2)\}\,\{1 - (m:3)\,(n:3)\}\,\{1 - (m:5)\,(n:5)\}$$

$$= \Pi\,\{1 - (m:p)\,(n:p)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

where $(m:p)$, $(n:p)$ have the meaning previously assigned to them, p denotes any prime number, and the product embraces all the primes (not reckoning unity among them).*

Then the symmetry of (13) shows that $P(m, n) = P(n, m)$, and since (see the footnote to § 1), we have $(-m:p) = (m:p)$ when m is any positive integer or zero, $P(-m, n) = P(m, n)$.

Thus, supposing m and n to have any positive integral or zero values, we see that

$$P(m,n)=P(n,m)=P(-m,n)=P(m,-n)=P(-m,-n) \ldots (14).$$

Every prime p which does not divide m makes $(m:p)$ vanish, and gives a unit factor in (13); but every prime divisor of m makes $(m:p) = 1$, and yields a factor $1 - (n:p)$.

Hence $$P(m, n) = \Pi\,\{1 - (n:p)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15),$$

where the p's which now occur in the product are the prime divisors of m.*

If n is not divisible by any prime divisor of m, we have $(n:p) = 0$ for all values of p, in which case every factor of (15) is unity. On the other hand (15) has a zero factor corresponding to each common prime divisor of m and n, when there are any. Consequently $P(m, n) = 1$ or 0 according as m, n are or are not prime to one another.

A separate consideration of the cases in which $m = 0, 1, n$ shows immediately that

$$P(1, n) = P(1, 1) = P(0, 1) = 1,$$

$$P(n, n) = P(0, n) = P(0, 0) = 0,$$

where n is any integer greater than 1.

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* It should be noticed that it is not necessary, but only convenient with a view to the future, to stipulate that p shall mean a prime in (13) and (15); p might have meant any number greater than 1 in (13), and any divisor of m except unity in (15), without affecting the reasoning of the present article.

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7. If $P(n)$

$$=\{1-(n:2)\}\{1-(n:3)\}\{1-(n:5)\}\ldots\{1-(n:p_\kappa)\}\ldots(16),$$

where p_κ (the κ^{th} prime) is so chosen that n is intermediate in value between p_κ and $p'_{\kappa+1}$, it is very easily seen that $P(n)=1$ or 0 according as n is a prime or composite number. In fact (16) is merely a representation, in the form of an equation, of the "sieve" of Eratosthenes.

Ex. If $p_\kappa=5$, $p_{\kappa+1}=7$, and

$$P(n)=\{1-(n:2)\}\{1-(n:3)\}\{1-(n:5)\}.$$

Then $P(n)=0$ when n is any composite number from 6 to 48 inclusive, and $P(n)=1$ when n is any prime number from 7 to 47 inclusive.

Writing $n=1, 2, 3, 4, \ldots, p_\kappa$ in (16), we find

$$P(1)=1, \text{ and } P(2)=P(3)=P(4)=\ldots=P(p_\kappa)=0,$$

so that

$$P(1)+P(2)+P(3)+P(4)+\ldots+P(n)=1+P(p_{\kappa+1})+\ldots+P(n)$$

i.e., this sum is the number of primes from p_κ to n (both included).

The value of p_κ in (16) is not fully determined, but it becomes so if we assume (as is usually done) that $p_{\kappa+1}$ is the *next* prime above the square root of n, in which case

$$P(1)+P(2)+P(3)+\ldots+P(n)$$

is one more than the number of primes superior to $\sqrt{n}$ in the natural sequence $1, 2, 3, \ldots, n$.

Expanding (16) we find

$$P(n)=1-\Sigma(n:p_\lambda)+\Sigma(n:p_\lambda p_\mu)-\Sigma(n:p_\lambda p_\mu p_\nu)+\ldots,$$

where the summations indicated by Σ extend to all values of $p_\lambda, p_\mu, p_\nu, \ldots$, which here denote all the primes from 2 to p_κ inclusive.

Hence, using equation (6) given *ante* in § 2, we obtain by addition

$$P(1)+P(2)+P(3)+\ldots$$

$$+P(n)=n-\Sigma\left[\frac{n}{p_\lambda}\right]+\Sigma\left[\frac{n}{p_\lambda p_\mu}\right]-\Sigma\left[\frac{n}{p_\lambda p_\mu p_\nu}\right]+\ldots.(17).$$

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A modification of this well-known result is found by starting from

$$P(n)=\{1-(n:2)\}\{1-(n:3)\}\{1-(n:4)\}\ldots\{1-(n:q)\}\ldots \quad (18),$$

which can be shown without difficulty to agree with (16) if we make q equal either to p_κ or to any one of the composite numbers between p_κ and $p_{\kappa+1}$; (the proof of this is suppressed to avoid prolixity). As a matter of convenience we make q equal to the integral part of the square root of n, which is equivalent to the assumption previously made respecting the value of p_κ.

Expanding (18), we obtain

$$P(n)=1-\Sigma(n:m_1)+\Sigma(n:m_2)-\Sigma(n:m_3)+\ldots\ldots(19),$$

where m_1 stands for any one of the natural numbers $2, 3, 4, \ldots, q$, the least common multiples of any combinations of $2, 3, \ldots$, of these numbers are denoted by $m_2, m_3, \ldots$, respectively, and the summations indicated by Σ extend to all the values of $m_1, m_2, m_3, \ldots$, respectively.

This expansion is a simple consequence of the obvious fact, that any number n is or is not divisible by both of the numbers r, s according as it is or is not divisible by their least common multiple $\mu_{r,s}$, which is expressed by the identity,

$$(n:r)(n:s)=(n:\mu_{r,s}) \quad \ldots\ldots\ldots\ldots\ldots(20).$$

From (19), by addition, we get

$$P(1)+P(2)+P(3)+\ldots$$

$$+P(n)=n-\Sigma\left[\frac{n}{m_1}\right]+\Sigma\left[\frac{n}{m_2}\right]-\Sigma\left[\frac{n}{m_3}\right]+\ldots\ldots(21),$$

and, as before,

$$P(1)=1,\ P(2)=P(3)=P(4)\ldots=P(q)=0,$$

so that (q being the integral part of $\sqrt{n}$), we still have

$$P(1)+P(2)+P(3)+P(4)+\ldots+P(n)$$

$$=1+P(q+1)+P(q+2)+\ldots+P(n),$$

= the number, increased by unity, of primes superior to $\sqrt{n}$ in the natural sequence $1, 2, 3, \ldots, n$.

As an example, to illustrate (17) and (21), let $n=23$; then $q=4$ and $p_\kappa=3$.

The primes from $p_{\kappa+1}$ to n (viz. 5, 7, 11, 13, 17, 19, 23) are 7 in number, and (17) gives

$$23-\left[\tfrac{23}{2}\right]-\left[\tfrac{23}{3}\right]+\left[\tfrac{23}{6}\right]=23-11-7+3=8.$$

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In (21) the values of m_1 are 2, 3, 4, those of m_2 are 6, 4, 12 (the least common multiples of 2, 3, of 2, 4, and of 3, 4, respectively), that of m_3 is 12 (the least common multiple of 2, 3, 4).

Thus 21 gives

$$23 - \left[\tfrac{23}{2}\right] - \left[\tfrac{23}{3}\right] - \left[\tfrac{23}{4}\right] + \left[\tfrac{23}{6}\right] + \left[\tfrac{23}{4}\right] + \left[\tfrac{23}{12}\right] - \left[\tfrac{23}{12}\right]$$
$$= 23 - \left[\tfrac{23}{2}\right] - \left[\tfrac{23}{3}\right] + \left[\tfrac{23}{6}\right] = 8 \text{ as before.}$$

8. The function $P(n)$ of the last article is a special form of the function $P(m, n)$, viz. the one wherein the prime divisors of m are all the primes from 2 to the integral part of $\sqrt{n}$.

We now resume the discussion of the general function $P(m, n)$.

In the first place, since $P(m, n) = 1$ or 0 according as m is prime to n or not, we have

$$P(1, n) + P(2, n) + P(3, n) + \ldots + P(n, n) = \tau(n) \ldots . (22),$$

$$1^r P(1, n) + 2^r P(2, n) + 3^r P(3, n) + \ldots + n^r P(n, n) = \Sigma t^r,$$

and more generally, ϕ being any arbitrary function,

$$\phi(1) P(1, n) + \phi(2) P(2, n) + \phi(3) P(3, n) + \ldots$$
$$+ \phi(n) P(n, n) = \Sigma \phi(t) \ldots \ldots \ldots \ldots \ldots (23),$$

where t means any one of the *totitives* of n (numbers not exceeding n and prime to it) and Σ refers to all of them.

Ex. gr. $$\log(1) P(1, n) + \log(2) P(2, n) + \log(3) P(3, n) + \ldots$$
$$+ \log(n) P(n, n) = \log(t_1, t_2, \ldots, t_{\tau(n)}),$$

where $t_1, t_2, \ldots, t_{\tau(n)}$ are the totitives of n.

From (15) we obtain

$$P(1, n) = 1,$$
$$P(2, n) = 1 - (n : 2),$$
$$P(3, n) = 1 - (n : 3),$$
$$P(4, n) = 1 - (n : 2),$$
$$P(5, n) = 1 - (n : 5),$$
$$P(6, n) = 1 - (n : 2) - (n : 3) + (n : 6),$$
$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots,$$

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and in general, just as in the preceding article,

$$P(m, n) = 1 - \Sigma(n : p_\lambda) + \Sigma(n : p_\lambda p_\mu) - \Sigma(n : p_\lambda p_\mu p_\nu) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (24),$$

where $p_\lambda, p_\mu, p_\nu, \ldots$, are the prime divisors of m.

Hence, multiplying the above equations by $\phi(1), \phi(2), \phi(3), \ldots$, and adding, we find

$$\begin{aligned}\Sigma\phi(t) = {} & \{\phi(1) + \phi(2) + \phi(3) + \ldots + \phi(n)\} \\ & - (n : 2)\{\phi(2) + \phi(4) + \phi(6) + \ldots + \phi(n)\} \\ & - (n : 3)\{\phi(3) + \phi(6) + \phi(9) + \ldots + \phi(n)\} \\ & - (n : 5)\{\phi(5) + \phi(10) + \phi(15) + \ldots + \phi(n)\} \\ & + (n : 6)\{\phi(6) + \phi(12) + \phi(18) + \ldots + \phi(n)\}, \\ & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

If we use $p_1, p_2, p_3, \ldots$ to denote the prime divisors of n, and if for shortness we write

$$\phi(d) + \phi(2d) + \phi(3d) + \ldots + \phi(n) = \psi(n, d),$$

(d being any divisor of n); the result just obtained takes the convenient form

$$\Sigma\phi(t) = \psi(n, 1) - \Sigma\psi(n, p_1) + \Sigma\psi(n, p_1p_2) - \Sigma\psi(n, p_1p_2p_3) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (24).$$

Ex. If $\phi(t) = 1$, we have $\Sigma\phi(t) = \tau(n)$ and $\psi(n, d) = \frac{n}{d}$, so that (24) gives

$$\tau(n) = n - \Sigma\frac{n}{p_1} + \Sigma\frac{n}{p_1p_2} - \Sigma\frac{n}{p_1p_2p_3} + \ldots,$$

which is the expanded form of

$$\tau(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\ldots.$$

9. When $\phi(t) = t^r$ a well-known formula gives

$$\psi(n, 1) = 1^r + 2^r + 3^r + \ldots + n^r = \text{const.} + \frac{n^{r+1}}{r+1} + \tfrac{1}{2}n^r + \tfrac{1}{12}rn^{r-1} - \ldots$$

$$= C + an^{r+1} + bn^r + c_1n^{r-2} - c_2n^{r-3} + c_3n^{r-5} - \ldots \text{(suppose)}.$$

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Changing n into $\frac{n}{d}$ and multiplying by d^r, we obtain

$$\psi(n, d) = Cd^r + a\frac{n^{r+1}}{d} + bn^r + c_1 n^{r-1}d - c_2 n^{r-2}d^2 + \ldots.$$

Substituting for each ψ in (24) its value, we find

$$\begin{aligned}\Sigma t^r = \quad & C(1 - \Sigma p_1^r + \Sigma p_1^r p_2^r - \Sigma p_1^r p_2^r p_3^r + \ldots) \\ & + an^{r+1}\left(1 - \Sigma\frac{1}{p_1} + \Sigma\frac{1}{p_1 p_2} - \Sigma\frac{1}{p_1 p_2 p_3} + \ldots\right) \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

To simplify this expression let the symbol $\theta_x(n)$ denote the function of x and n defined by the equation

$$\theta_x(n) = n^{-x}(1 - p_1^x)(1 - p_2^x)(1 - p_3^x)\ldots\ldots \quad (25);$$

then remembering that $\theta_{-1}(n) = \tau(n)$, the above result becomes

$$\Sigma t^r = n^r\{C\theta_r(n) + a\tau(n) + b\theta_0(n) + c_1\theta_1(n) - c_2\theta_2(n) + \ldots\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (26).$$

In particular, giving r the values 0, 1, 2, 3, 4, 5, we obtain

$$\Sigma t^0 = \tau(n),$$
$$\Sigma t = \tfrac{1}{2}n\{\tau(n) + \theta_0(n)\},$$
$$\Sigma t^2 = n^2\{\tfrac{1}{3}\tau(n) + \tfrac{1}{2}\theta_0(n) + \tfrac{1}{6}\theta_1(n)\},$$
$$\Sigma t^3 = n^3\{\tfrac{1}{4}\tau(n) + \tfrac{1}{2}\theta_0(n) + \tfrac{1}{4}\theta_1(n)\},$$
$$\Sigma t^4 = n^4\{\tfrac{1}{5}\tau(n) + \tfrac{1}{2}\theta_0(n) + \tfrac{1}{3}\theta_1(n) - \tfrac{1}{30}\theta_3(n)\},$$
$$\Sigma t^5 = n^5\{\tfrac{1}{6}\tau(n) + \tfrac{1}{2}\theta_0(n) + \tfrac{5}{12}\theta_1(n) - \tfrac{1}{12}\theta_3(n)\}.$$

In these formulae, we may in general (when n is any any integer greater than 1) make $\theta_0(n) = 0$, but when $n = 1$, we must take $\theta_0(n) = 1$. For equation (25), which defines the function $\theta_x(n)$, shows that $\theta_x(1) = 1$ for all values of x including $x = 0$; but when n has any prime divisor p_1 (which will be the case when $n > 1$) the function $\theta_0(n)$ contains the zero factor $1 - p_1^0$. This explains the fact that the substitution of 1 for n in the general expression, $\frac{1}{2}n\tau(n)$, for the sum of the totitives of n would give $\frac{1}{2}$ for the sum of the totitives of 1. The true value is, as we have seen,

$$\Sigma t = \tfrac{1}{2}n\{\tau(n) + \theta_0(n)\},$$

where $\theta_0(n) = 0$ or 1 according as n is greater than or equal to 1.

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10. Let ϕ be any rational integral function of its argument, say

$$\phi(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_l t^l.$$

Then $\Sigma\phi(t)$ is finitely expressible in terms of n and its prime divisors, by a very simple process, as follows.

Write $\phi_{-1}(n) = a_0 n + \frac{1}{2} a_1 n^2 + \frac{1}{3} a_2 n^3 + \frac{1}{4} a_3 n^4 + \ldots + \frac{a_l}{l+1} n^{l+1}$,

$$\phi_1(n) = 2a_2 n + 3a_3 n^2 + \ldots + l a_l n^{l-1},$$

and so on; where ϕ_{-1} is obtained from ϕ by integration between the limits 0 and n, ϕ_1 is derived from ϕ by a single differentiation followed by the removal of the constant term a_1, and so in general ϕ_κ is derived from ϕ by differentiating it κ times and removing the constant term which appears when the last differentiation is performed. The last of these functions will be ϕ_{l-1}, for $\phi_l = 0$ owing to the removal of the constant term.

Multiply the successive values of Σt^r which correspond to $r = 0, 1, 2, 3, \ldots, l$ by $a_0, a_1, a_2, a_3, \ldots, a_l$ respectively, and add.

When n is any integer greater than 1, the result will be

$$\Sigma\phi(t) = \frac{\tau(n)}{n}\phi_{-1}(n) + \tfrac{1}{12} n \theta_1(n)\phi_1(n) - \tfrac{1}{720} n^3 \theta_3(n)\phi_3(n) + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (27),$$

where the functions ϕ_{-1}, ϕ_1, ϕ_3, ... are as defined above. when $n = 1$, we have $\Sigma t^r = 1$ for all values of r, and consequently $\Sigma\phi(t) = \phi(1)$.

Writing $\tau(n) = \theta_{-1}(n)$ for the sake of uniformity, equation (27) may be put under the following simple symbolical form

$$\Sigma\phi(t) = \tfrac{1}{2}\coth(\tfrac{1}{2}z) \quad \ldots\ldots\ldots\ldots\ldots (28).$$

where, after expanding in ascending powers of z, we replace z^r by $n^r \theta_r(n)\phi_r(n)$, or what, is the same thing, by

$$\phi_r(n)(1 - p_1^r)(1 - p_2^r)(1 - p_3^r)\ldots.$$

The proof is quite easy; for on substituting for $a, b, c_1, c_2, \ldots$ their known values, and writing $D^{-1}n^r$, Dn^r, ... in place of $\frac{1}{r+1} n^{r+1}$, $r n^{r-1}$, ..., equation (26) gives

$$\Sigma t^r = \{ C n^r \theta_r(n) \frac{D^r}{r!} + n^{-1}\theta_{-1}(n) D^{-1}$$

$$+ \tfrac{1}{2}\theta_0(n) + \tfrac{1}{12} n \theta_1(n) D - \tfrac{1}{720} n^3 \theta_3(n) D^3 + \ldots \} n^r$$

$$= \{ C\kappa^r \frac{D^r}{r!} + \kappa^{-1} D^{-1} \tfrac{1}{12}\kappa D - \tfrac{1}{720}\kappa^3 D^3 + \ldots \} n^r,$$

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where $\theta_0(n)$ has been put $=0$ (because we are considering the case of $n>1$) and any power κ^s of κ stands symbolically for $n^s\theta_s(n)$.

Now $\kappa^{-1}D^{-1}+\frac{1}{12}\kappa D-\frac{1}{720}\kappa^3D^3+\ldots$

$$=(e^{\kappa D}-1)^{-1}+\tfrac{1}{2}=\tfrac{1}{2}\left(\frac{e^{\kappa D}+1}{e^{\kappa D}-1}\right)=\tfrac{1}{2}\coth(\tfrac{1}{2}\kappa D),$$

so that
$$\Sigma t^r=\{C\kappa^r\frac{D^r}{r!}+\tfrac{1}{2}\coth(\tfrac{1}{2}\kappa D)\}\,n^r \quad \ldots\ldots\ldots (29),$$

and, were it not for the presence of the C, we should have

$$\Sigma\phi(t)=\tfrac{1}{2}\coth(\tfrac{1}{2}\kappa D)\,\phi(n).$$

But the work of Art. 9 shows that the value of C in Σt^r is the same as in

$$1^r+2^r+3^r+\ldots+n^r=C+an^{r+1}+\ldots,$$

and it is easily seen that (in the case under consideration, that of r a positive integer) the correct value of C makes the coefficient of κ^rD^r in (29) vanish. This leads to the rule given above of removing the constant term from the successive derivatives of $\phi(n)$, so that our result becomes

$$\Sigma\phi(t)=\tfrac{1}{2}\coth(\tfrac{1}{2}\kappa D)\,\phi(n),$$

where the successive derivatives of $\phi(n)$ have to be replaced by $\phi_1(n)$, $\phi_2(n)$, ...; which agrees with (28).

11. Nearly all the properties of $\tau(n)$, if not all of them, may be extended to the function $\theta_x(n)$. Thus *ex. gr.* if m, n are any two numbers and $\Delta_{m,n}$ their greatest common measure, we have

$$\theta_x(mn)=\theta_x(m)\,\theta_x(n)\,\frac{\Delta^{-x}_{m,n}}{\theta_x(\Delta_{m,n})} \quad \ldots\ldots\ldots (30),$$

which corresponds to

$$\tau(mn)=\tau(m)\,\tau(n)\,\frac{\Delta_{m,n}}{\tau(\Delta_{m,n})}.$$

To prove this, let

$$\theta_x(m)=m^{-x}\,\Pi(1-p^x)(1-q^x),$$
$$\theta_x(n)=n^{-x}\,\Pi(1-q^x)(1-r^x),$$

where p is any prime divisor of m only, r any prime divisor of n only, and q any prime divisor of both m and n.

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Then $$\theta_x(mn) = m^{-x}n^{-x}\Pi(1-p^x)(1-q^x)\ (1-r^x),$$

$$\theta_x(m)\,\theta_x(n) = m^{-x}n^{-x}\Pi(1-p^x)(1-q^x)^2(1-r^x),$$

and $$\theta_x(\Delta_{m,n}) = \Delta^{-x}_{m,n}\Pi(1-q^x).$$

Hence $$\theta_x(mn)\,\theta_x(\Delta_{m,n}) = \Delta^{-x}_{m,n}m^{-x}n^{-x}\Pi(1-p^x)(1-q^x)^2(1-r^x)$$

$$= \Delta^{-x}_{m,n}\,\theta_x(m)\,\theta_x(n),$$

which proves (30).

The formula

$$(n:1)\,\theta_x(1) + (n:2)\,\theta_x(2) + (n:3)\,\theta_x(3) + \ldots = n^{-x} \ldots (31)$$

corresponding to

$$(n:1)\,\tau(1) + (n:2)\,\tau(2) + (n:3)\,\tau(3) + \ldots = n$$

may, like it, be established in several ways. One proof is given by Dr. Glaisher in the *Phil. Mag.* for December, 1884 (p. 531), in a paper entitled "Applications of Möbius's Theorem on the Reversion of Certain Series" which also contains (p. 537) the determinant for $\tau(n)$ numbered (9) in Art. 3 of this paper, and (p. 538) the precisely similar determinant for $\theta_x(n)$, obtainable from (9) by changing the natural numbers in its last column into $1^{-x}, 2^{-x}, 3^{-x}, \ldots$.

The determinant (10) is, as we have stated, deducible from the formula

$$\left[\frac{n}{1}\right]\tau(1) + \left[\frac{n}{2}\right]\tau(2) + \left[\frac{n}{3}\right]\tau(3) + \ldots = \tfrac{1}{2}n(n+1),$$

first given, we believe, by Sylvester, many years ago; and a determinant for $\theta_x(n)$ similar to (10) but with the triangle numbers in its last column replaced by 1^{-x}, $1^{-x}+2^{-x}$, $1^{-x}+2^{-x}+3^{-x}, \ldots$, is deducible from the formula

$$\left[\frac{n}{1}\right]\theta_x(1) + \left[\frac{n}{2}\right]\theta_x(2) + \left[\frac{n}{3}\right]\theta_x(3) + \ldots = 1^{-x} + 2^{-x} + 3^{-x} + \ldots + n^{-x}$$

got from (31) by addition.

In conclusion we remark that formulæ involving the symbol $(m:n)$ are instantaneously transformed into generating function formulæ.

For example, (31) gives immediately

$$\frac{\theta_x(1)\,q}{1-q} + \frac{\theta_x(2)\,q^2}{1-q^2} + \frac{\theta_x(3)\,q^3}{1-q^3} + \ldots = 1^{-x}q + 2^{-x}q^2 + 3^{-x}q^3 + \ldots,$$

where in transforming we have only to remember that the coefficient of q^n in $\dfrac{q^m}{1-q^m}$ is $(n:m)$.

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ON THE ELLIPTIC FUNCTIONS OF $\frac{1}{3}K$.

By *J. W. L. Glaisher.*

§ 1. In Vol. XII of the *Messenger* (pp. 134 and 154) there are two papers by Dr. Forsyth and Mr Burnside on this subject. In the former of these papers Dr. Forsyth obtained for $\operatorname{sn}\frac{1}{3}K$ the value

$$\frac{2}{1+(1-c)^{\frac{1}{2}}+\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\}^{\frac{1}{2}}},$$

where $c^3=4k^2k'^2$, the sign of the radical $(1-c)^{\frac{1}{2}}$ having been determined by putting $k=0$.

In the second paper Mr. Burnside gave a very interesting system of values for $\operatorname{sn}^2\frac{4}{3}K$, $\operatorname{sn}^2\frac{4}{3}iK'$, &c. These formulæ lead directly to the following results:

$$\operatorname{cd}^2\tfrac{1}{3}K=\frac{1}{k}\left[\sqrt{\{2\,(1-\lambda+\lambda^2)^{\frac{1}{2}}+2-\lambda\}}-\sqrt{(1+\lambda)}\right],$$

$$\operatorname{cn}^2\tfrac{1}{3}K=\frac{k'}{k}\left[\sqrt{\{2\,(1+\mu+\mu^2)^{\frac{1}{2}}-2-\mu\}}\mp\sqrt{(\mu-1)}\right],$$

$$\operatorname{cs}^2\tfrac{1}{3}K=k'\left[\sqrt{\{2\,(1-\nu+\nu^2)^{\frac{1}{2}}+2-\nu\}}+\sqrt{(1+\nu)}\right],$$

where $\lambda=\left(\frac{k'^2}{2k}\right)^{\frac{2}{3}}$, $\mu=\left(\frac{1}{2kk'}\right)^{\frac{2}{3}}$, $\nu=\left(\frac{k^2}{2k'}\right)^{\frac{2}{3}}$.

In the formula for $\operatorname{cn}^2\frac{1}{3}K$ the upper or lower sign is to be taken according as $k<$ or $>k'$.

§ 2. From these values of $\operatorname{cd}^2\frac{1}{3}K$, &c. I have deduced the following system of formulæ for six of the elliptic functions of $\frac{1}{3}K$:—

$$\operatorname{sn}\tfrac{1}{3}K=\tfrac{1}{2}\left[1+\sqrt{\{2\,(1-u+u^2)^{\frac{1}{2}}+2-u\}}-\sqrt{(1+u)}\right],$$

$$k'\operatorname{sd}\tfrac{1}{3}K=\tfrac{1}{2}\left[1-\sqrt{\{2\,(1-u+u^2)^{\frac{1}{2}}+2-u\}}+\sqrt{(1+u)}\right],$$

$$\operatorname{ns}\tfrac{1}{3}K=\tfrac{1}{2}\left[\;1+\sqrt{\{2\,(1+w+w^2)^{\frac{1}{2}}+2+w\}}\pm\sqrt{(1-w)}\right],$$

$$k'\operatorname{nd}\tfrac{1}{3}K=\tfrac{1}{2}\left[-1+\sqrt{\{2\,(1+w+w^2)^{\frac{1}{2}}+2+w\}}\pm\sqrt{(1-w)}\right],$$

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$$\frac{1}{k'}\operatorname{ds}\tfrac{1}{3}K=\tfrac{1}{2}\left[\ 1+\sqrt{\{2\,(1-v+v^2)^{\frac{1}{2}}+2-v\}}+\sqrt{(1+v)}\right],$$

$$\frac{1}{k'}\operatorname{dn}\tfrac{1}{3}K=\tfrac{1}{2}\left[-1+\sqrt{\{2\,(1-v+v^2)^{\frac{1}{2}}+2-v\}}+\sqrt{(1+v)}\right],$$

where $u=\left(\frac{2k'}{k^2}\right)^{\frac{1}{3}}$, $w=(2kk')^{\frac{1}{3}}$, $v=\left(\frac{2k}{k'^2}\right)^{\frac{1}{3}}$.

In the two middle formulæ, the upper or lower sign is to be taken according as $k<$ or $>k'$.

§ 3. Dr. Forsyth's value, which may be written

$$2\operatorname{ns}\tfrac{1}{3}K=1+(1-c)^{\frac{1}{3}}+\{2+c+2\,(1+c+c^2)^{\frac{1}{2}}\},$$

is the same as the third of the above relations, when $k<k'$.

§ 4. The method by which the above results were derived from Mr. Burnside's formulæ is given in §§ 19–23 of a paper "On the q-series derived from the Elliptic and Zeta Functions of $\frac{1}{3}K$ and $\frac{1}{4}K$," which was read before the London Mathematical Society on December 11, 1890. As the investigation and results occupy a subordinate place in that paper, I have ventured to reproduce the latter in this note in order that they might appear in the same periodical as Dr. Forsyth's and Mr. Burnside's values. It would be interesting to obtain the system of values for the other six elliptic functions of $\frac{1}{3}K$, to which my method did not lead me.

I may add that the same paper also contains the values of ρ_3, $k_3\rho_3$, $k_3^{\frac{1}{3}}\rho_3$, $k'_3\rho$, ρ_3^2, &c., and I_3, G_3, E_3, &c., these being the quantities with which ρ, $k\rho$, &c., and I, G, E, &c. are transformed by the change of q into q^3.

END OF VOL. XX.

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The following papers have been received:

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